

Fibonacci numbers in nature

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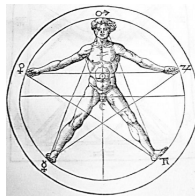
What are the Fibonacci numbers?

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0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

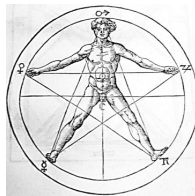
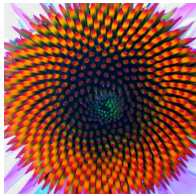
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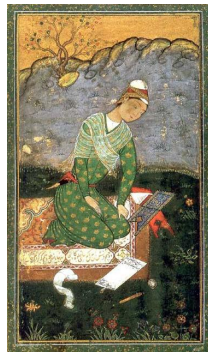
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...



One of these is **not** exactly related to the Fibonacci numbers.

A little history

- Studied in India as early as 200 BC



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- Introduced to the West by **Leonardo of Pisa** (Fibonacci) in *Liber Abaci* (1202)



Leonardo of Pisa
c. 1170 – c. 1250

A little history

- Studied in India as early as 200 BC
- Introduced to the West by Leonardo of Pisa (Fibonacci) in *Liber Abaci* (1202)
 - “Book of Calculation”
 - Described Hindu-Arabic numerals
 - Used Fibonacci numbers to model rabbit population growth



Leonardo of Pisa
c. 1170 – c. 1250

Bunnies!

Model assumptions

- One male-female pair originally
- Each pair able to mate at one month, mating each month thereafter
- Each mating produces one new pair after one month



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How many pairs are there after n months?

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1									

Month 1

- One pair originally

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1								

Month 2

- From last month: 1
- Newly born: 0

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2							

Month 3

- From last month: 1
- Newly born: 1

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3						

Month 4

- From last month: 2
- Newly born: 1

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5					

Month 5

- From last month: 3
- Newly born: 2

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8				

Month 6

- From last month: 5
- Newly born: 3

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8	13			

Month 7

- From last month: 8
- Newly born: 5

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8	13	21		

Month 8

- From last month: 13
- Newly born: 8

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8	13	21	...	F_n

Month n

- From last month: F_{n-1}
- Newly born: F_{n-2}

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8	13	21	...	F_n

$$F_1 = F_2 = 1 \quad \text{(seed values)}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{(recurrence relation)}$$

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
pairs	1	1	2	3	5	8	13	21	...	F_n

$F_0 = 0, \quad F_1 = 1$ (seed values)

$F_n = F_{n-1} + F_{n-2}$ (recurrence relation)

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	...	n
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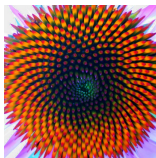
$$F_0 = 0, \quad F_1 = 1 \quad \text{(seed values)}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{(recurrence relation)}$$

Note that the rabbit model is **unrealistic** (why?), but we will see a real instance where the Fibonacci numbers show up very shortly.

Fibonacci numbers in nature

n	0	1	2	3	4	5	6	7	8	9	10	11	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	...



Fibonacci numbers in nature

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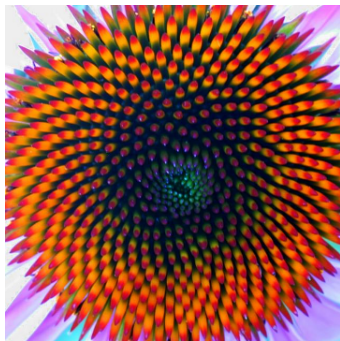


Number of spirals

Clockwise:	13
Counterclockwise:	8

Fibonacci numbers in nature

n	0	1	2	3	4	5	6	7	8	9	10	11	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	...



Number of spirals

Clockwise:	21
Counterclockwise:	34

Fibonacci numbers in nature

n	0	1	2	3	4	5	6	7	8	9	10	11	...
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n	0	1	2	3	4	5	6	7	8	9	10	11	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	...



So where do the Fibonacci numbers come from?

A crash course on plant growth

- Central turning growing tip
- Emits new seed head, floret, leaf bud, etc. every α turns
- Seed heads grow outward with time



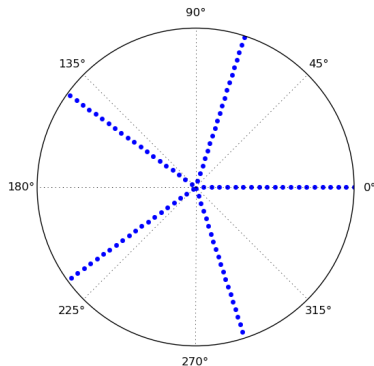
A crash course on plant growth

$$\alpha = 1/4$$

$$\alpha = 1/5$$

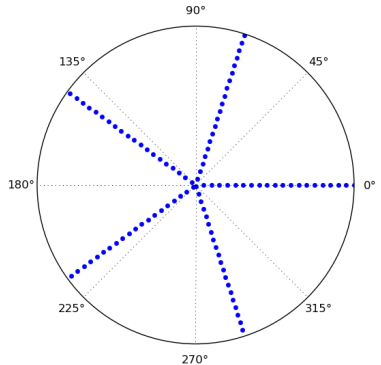
From a plant's perspective

- What's wrong with this growth pattern?

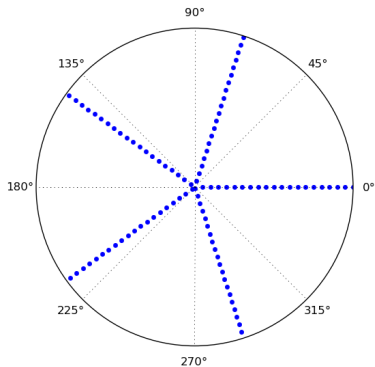


From a plant's perspective

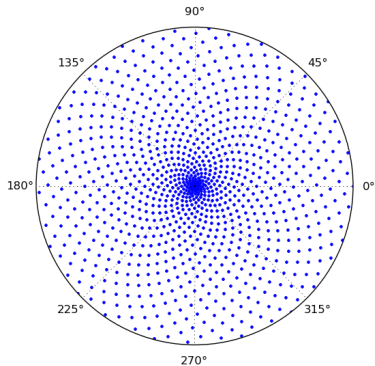
- What's wrong with this growth pattern?
- Too much **wasted space!**



From a plant's perspective

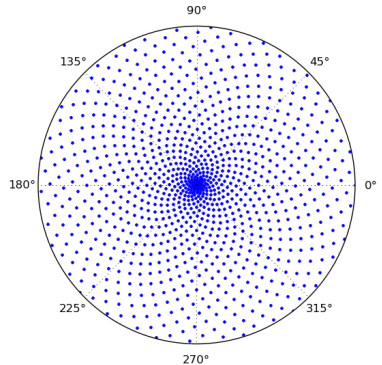


From a plant's perspective



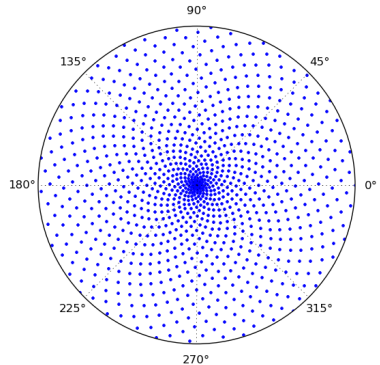
From a plant's perspective

- What's wrong with this growth pattern?
- Too much wasted space!
- Want to **maximize exposure** to sunlight, dew, CO₂



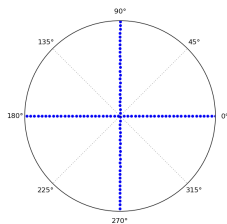
From a plant's perspective

- What's wrong with this growth pattern?
- Too much wasted space!
- Want to maximize exposure to sunlight, dew, CO₂
- Evolve for **optimal packing**

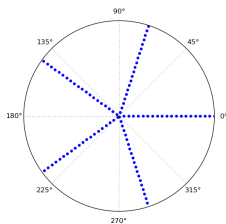


Floral showcase

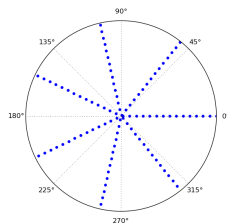
$$\alpha = 1/4$$



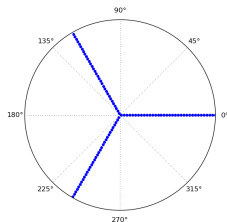
$$\alpha = 1/5$$



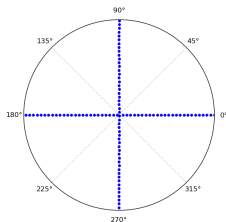
$$\alpha = 1/7$$



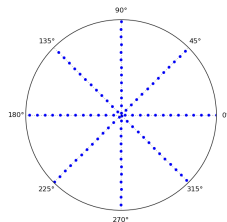
$$\alpha = 2/3$$



$$\alpha = 3/4$$



$$\alpha = 5/8$$



Rationality is not always good

Definition

A **rational number** is a number that can be expressed as a fraction m/n , where m and n are integers.

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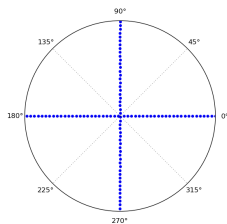
Can we get a good covering with $\alpha = m/n$? The answer is **no**.

Why?

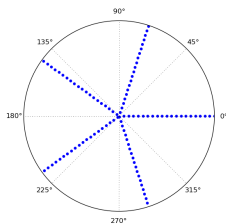
- Growing tip makes m revolutions every n seeds
- Growth pattern repeats after n seeds
- At most n “rays” of seeds

Floral showcase redux (rational)

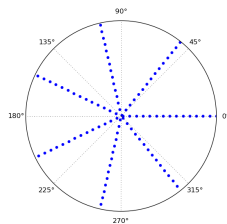
$$\alpha = 1/4$$



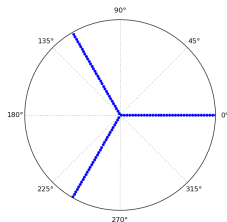
$$\alpha = 1/5$$



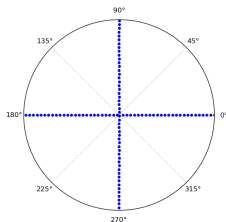
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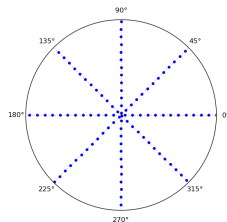
$$\alpha = 2/3$$



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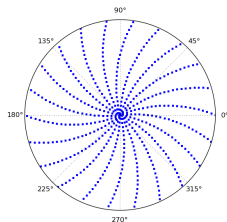


$$\alpha = 5/8$$

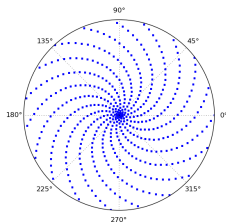


Floral showcase (irrational)

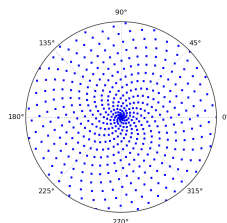
$$\alpha = 1/\pi$$



$$\alpha = 1/e$$

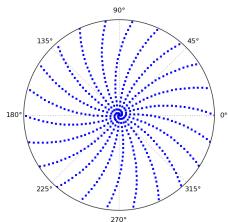


$$\alpha = 1/\sqrt{2}$$



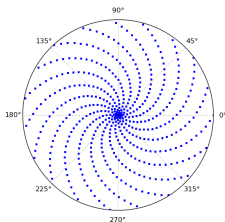
Floral showcase (irrational)

$$\alpha = 1/\pi$$

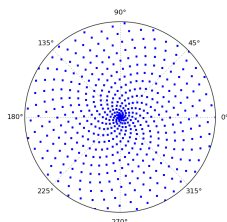


“Less” irrational

$$\alpha = 1/e$$



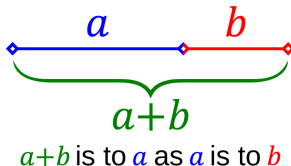
$$\alpha = 1/\sqrt{2}$$



“More” irrational

- Some irrationals work better than others.
- What is the “most” irrational number?

The golden ratio



Mathematically,

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi.$$

How to solve for φ ?

The golden ratio

1 Given:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi$$

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4 Rearrange:

$$\varphi^2 - \varphi - 1 = 0$$

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5 Quadratic formula:

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

The golden ratio

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$$\varphi^2 - \varphi - 1 = 0$$

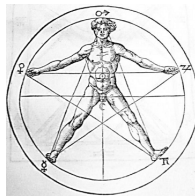
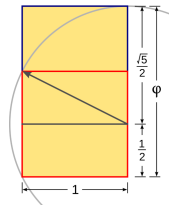
5 Quadratic formula:

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

The number $\varphi \approx 1.618\dots$ is called the **golden ratio**.

The golden ratio: a broader perspective

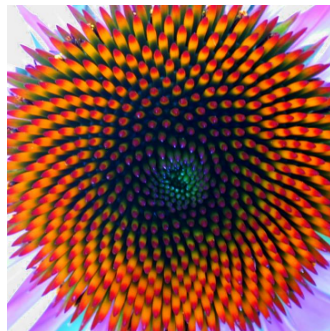
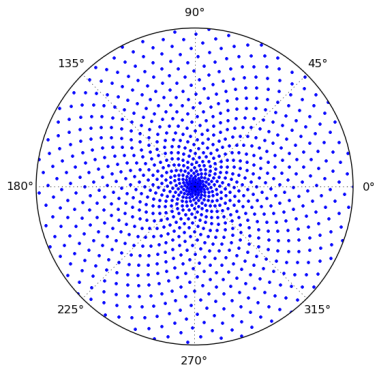
- Studied since antiquity
- First defined by Euclid (*Elements*, c. 300 BC)
- Associated with perceptions of beauty
- Applications in art and architecture



The golden ratio in plant growth

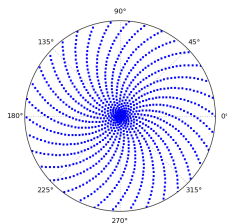
$$\alpha = 1/\varphi$$

The golden ratio in plant growth

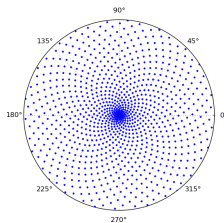


The golden ratio in plant growth

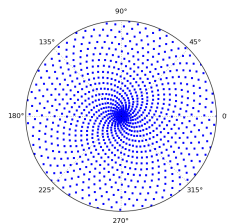
$$\alpha = 222.4^\circ$$



$$\alpha = 1/\varphi \approx 222.5^\circ$$



$$\alpha = 222.6^\circ$$



Nature seems to have found φ quite precisely!

Some properties of irrational numbers

Theorem

Every irrational number can be written as a **continued fraction**

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}$$

or, for short, $[a_0; a_1, a_2, \dots]$, where the a_i are positive integers.

Some properties of irrational numbers

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$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, \dots]$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots]$$

$$\sqrt{2} = [1; 2, 2, 2, \dots]$$

$$\varphi = [1; 1, 1, 1, \dots]$$

Some properties of irrational numbers

Theorem

Every irrational number can be written as a continued fraction $[a_0; a_1, a_2, \dots]$, where the a_i are positive integers.

The truncations

$$[a_0] = \frac{a_0}{1}, \quad [a_0; a_1] = \frac{a_1 a_0 + 1}{a_1},$$
$$[a_0; a_1, a_2] = \frac{a_2 (a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}, \quad \dots$$

give a sequence of rational approximations called **convergents**.

Some properties of irrational numbers

Theorem

The convergent $[a_0; a_1, a_2, \dots, a_k] \equiv m/n$ provides the **best approximation** among all rationals m'/n' with $n' \leq n$.

The truncations

$$\begin{aligned} [a_0] &= \frac{a_0}{1}, & [a_0; a_1] &= \frac{a_1 a_0 + 1}{a_1}, \\ [a_0; a_1, a_2] &= \frac{a_2 (a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}, & \dots \end{aligned}$$

give a sequence of rational approximations called convergents.

The most irrational number

A few convergents:

$$\blacksquare \pi: \quad 3, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}$$

$$\blacksquare e: \quad 2, \quad 3, \quad \frac{8}{3}, \quad \frac{11}{4}, \quad \frac{19}{7}$$

$$\blacksquare \sqrt{2}: \quad 1, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}$$

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$$22/7 \approx 3.14285714 \dots$$

$$333/106 \approx 3.14150943 \dots$$

$$355/113 \approx 3.14159292 \dots$$

The most irrational number

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$$\blacksquare \sqrt{2}: \quad 1, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}$$

$$22/7 \approx 3.14\textcolor{red}{285714} \dots$$

$$333/106 \approx 3.1415\textcolor{red}{0943} \dots$$

$$355/113 \approx 3.141592\textcolor{red}{92} \dots$$

What makes a number **easy** to approximate rationally?

The most irrational number

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$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ddots}}}}$$

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Large denominators mean small numbers!

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$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}}}$$

The golden ratio has the **slowest converging** representation.

The most irrational number

A few convergents:

$$\blacksquare \pi: \quad 3, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}$$

$$\blacksquare e: \quad 2, \quad 3, \quad \frac{8}{3}, \quad \frac{11}{4}, \quad \frac{19}{7}$$

$$\blacksquare \sqrt{2}: \quad 1, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}$$

$$\blacksquare \varphi: \quad 1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \frac{13}{8}$$

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}}}$$

The golden ratio has the slowest converging representation.

Back to the Fibonacci numbers

Theorem

The ratio of successive Fibonacci numbers $F_{n+1}/F_n \rightarrow \varphi$ as $n \rightarrow \infty$.

Back to the Fibonacci numbers

Theorem (in English)

The ratio of successive Fibonacci numbers $F_{n+1}/F_n \approx \varphi$, and the approximation gets better the bigger n is.

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Informally:

1 Exponential growth:

$$F_{n+1}/F_n \approx \theta$$

2 Recurrence relation:

$$F_n = F_{n-1} + F_{n-2}$$

3 Divide and rewrite:

$$\frac{F_n}{F_{n-1}} \frac{F_{n-1}}{F_{n-2}} = \frac{F_{n-1}}{F_{n-2}} + 1$$

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$$\theta^2 \approx \theta + 1$$

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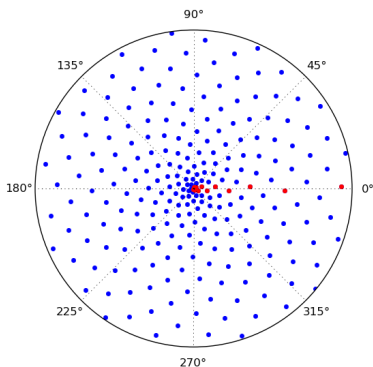
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This is just the equation for the **golden ratio**, so $\theta \approx \varphi$.

Going around

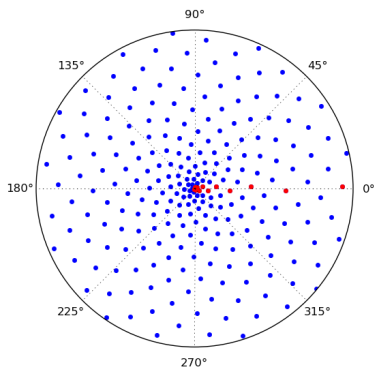
$$\alpha = 1/\varphi \approx F_n/F_{n+1}$$



- F_n revolutions over F_{n+1} seeds
- No exact repeat since irrational
- Alternately overshoot and undershoot

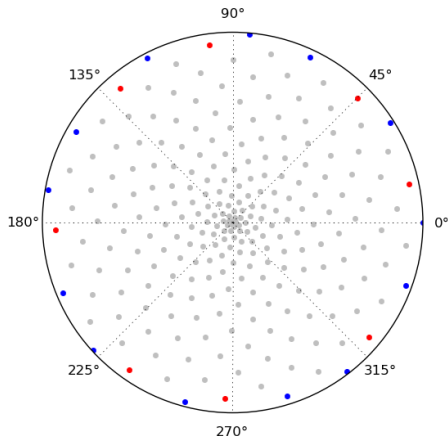
Going around

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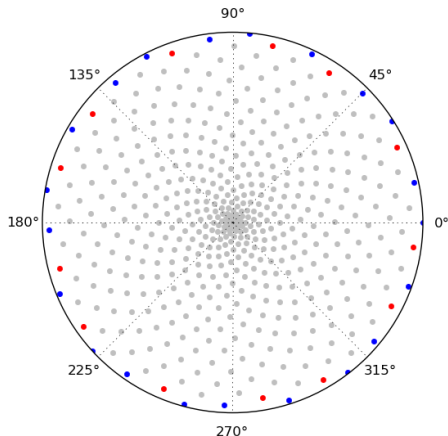
seeds	position
2	+0.236068
3	−0.145898
5	+0.090170
8	−0.055728
13	+0.034442
21	−0.021286
34	+0.013156
55	−0.008131
89	+0.005025
144	−0.003106

Origin of the spirals



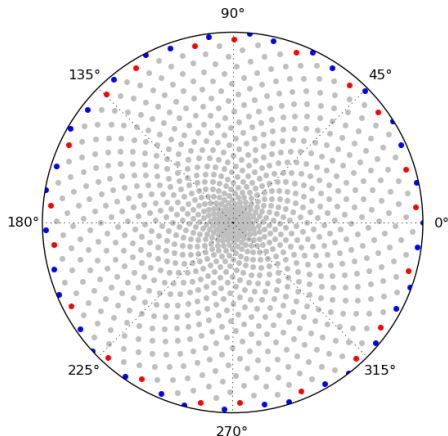
■ Seed heads:	250
■ CW spirals:	13
■ CCW spirals:	13 + 8

Origin of the spirals



- Seed heads: 500
- CW spirals: 21 + 13
- CCW spirals: 21

Origin of the spirals



- Seed heads: 1000
- CW spirals: 34
- CCW spirals: 34 + 21

Origin of the spirals

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Final note

There is a **very good** reason why the Fibonacci numbers show up in at least one aspect of nature (plant growth)—and now you know what it is. (Spread the word!)

Questions?



MoMA (Sep 2008)