


Fast direct methods for structured matrices

Kenneth L. Ho (Stanford)


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Introduction

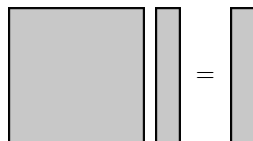
$$Ax = b$$


The diagram illustrates the matrix equation $Ax = b$. It features three gray rectangular blocks with black outlines. On the left is a large square representing the matrix A . To its right is a tall, narrow vertical rectangle representing the vector x . An equals sign is positioned between these two blocks. To the right of the equals sign is another tall, narrow vertical rectangle representing the vector b . The equation $Ax = b$ is written in black text above the diagram.


Introduction

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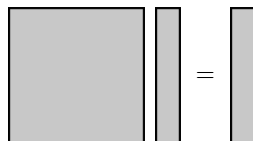
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- ▶ Classical methods infeasible beyond $N \sim 10^4$

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- ▶ Other common matrix problems:
 - $y = Ax$: $O(N^2)$
 - $A = UV^*$: $O(N^3)$
 - $\Delta = \det A$: $O(N^3)$

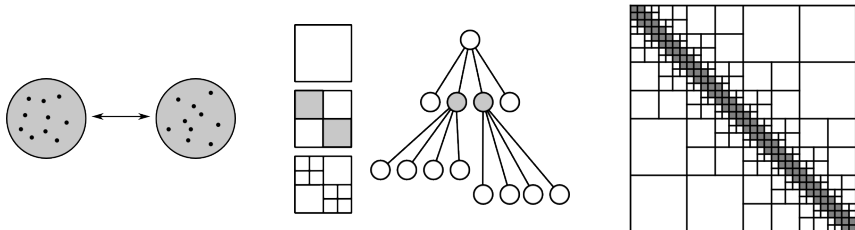
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- ▶ Observation: many matrices arising in practice are **structured**

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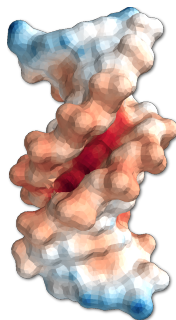
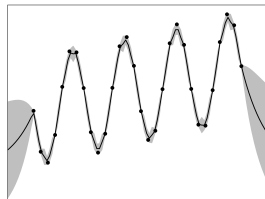
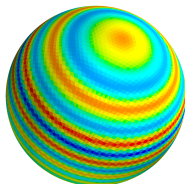
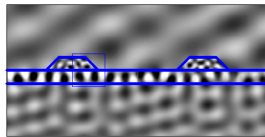
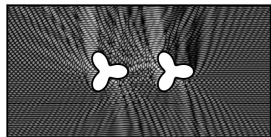
- ▶ For $A \in \mathbb{C}^{N \times N}$ dense, solution generally requires $O(N^3)$ work $\rightarrow O(N)$
- ▶ Classical methods infeasible beyond $N \sim 10^4$
- ▶ Other common matrix problems:
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 - $A = UV^*$: $O(N^3) \rightarrow O(N)$
 - $\Delta = \det A$: $O(N^3) \rightarrow O(N)$
- ▶ Observation: many matrices arising in practice are structured
- ▶ Goal: accelerate to **linear complexity** by exploiting matrix structure

- ▶ **Hierarchical matrices:** low-rank submatrices at a hierarchy of scales
- ▶ Canonical example: N -body problem
 - Particle locations: $x_i, i = 1, \dots, N$
 - Interaction kernel: $K(x, y) = 1/\|x - y\|$
 - Forces: $f_i = \sum_{j=1}^N K(x_i, x_j) m_j$
- ▶ Matrix $A_{ij} = K(x_i, x_j)$ can be applied in $O(N)$ time



Introduction

- Applications: integral equations, elliptic PDEs, machine learning, etc.



Introduction

Many structured matrix problems can be solved efficiently by iteration

- ▶ CG/GMRES + fast multiplication: $O(n_{\text{iter}}N)$ complexity
- ▶ Very successful; industrial applications in electromagnetics, acoustics, etc.

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But ...

- ▶ What if n_{iter} is large (high contrasts, geometric singularities, ill-conditioning)?
- ▶ What if there are many RHS's (time stepping, inverse problems)?

Compare with **direct** solvers: no convergence issues, efficient information reuse.

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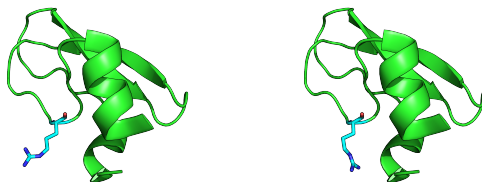
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Compare with direct solvers: no convergence issues, efficient information reuse.

In certain important environments, there is a need for fast direct methods.

Example: protein design

- ▶ Protein defined by a fixed backbone with flexible residue sidechains
- ▶ Each sidechain can be one of several rotamers $r_i \in R_i$
- ▶ Energy $E(\mathbf{r})$ depends on the joint rotamer configuration \mathbf{r}
- ▶ Goal: find \mathbf{r} such that $E(\mathbf{r})$ is **minimized**



- ▶ NP-hard but various strategies are available
- ▶ One of many related formulations

Example: protein design

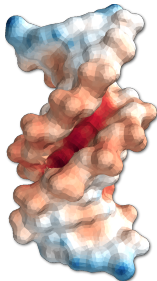
- ▶ Simplest approach: pairwise approximation

$$E(\mathbf{r}) \approx \sum_i E(r_i) + \frac{1}{2} \sum_i \sum_{j \neq i} E(r_i, r_j)$$

- ▶ Number of energy evaluations: $O((n_{\text{rot}} N_{\text{res}})^2)$
- ▶ Each evaluation requires a PDE solve for the electrostatic energy:

$$A_i x_i = b_i, \quad i = 1, \dots, O((n_{\text{rot}} N_{\text{res}})^2)$$

- ▶ Matrices A_i are perturbations of fixed backbone matrix A_0
- ▶ Precompute A_0^{-1} , rapid update for each $x_i = A_i^{-1} b_i$



Potential for massive acceleration using fast direct methods.

- ▶ **This talk:** our recent work on **fast direct methods** for structured matrices
- ▶ Many other contributors (apologies for an incomplete list)
- ▶ Focus on integral equations in 2D/3D, complex geometry
- ▶ Main result: **linear-complexity** generalized LU decomposition
- ▶ Sparsification/elimination + recursive dimensional reduction

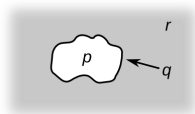
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Tools: sparse elimination, interpolative decomposition, **skeletonization**

Sparse elimination

Let

$$A = \begin{bmatrix} A_{pp} & A_{pq} \\ A_{qp} & A_{qq} & A_{qr} \\ & A_{rq} & A_{rr} \end{bmatrix}.$$



(Think of A as a **sparse** matrix.) If A_{pp} is nonsingular, define

$$R_p^* = \begin{bmatrix} I & & \\ -A_{qp}A_{pp}^{-1} & I & \\ & & I \end{bmatrix}, \quad S_p = \begin{bmatrix} I & -A_{pp}^{-1}A_{pq} & \\ & I & \\ & & I \end{bmatrix}$$

so that

$$R_p^* A S_p = \begin{bmatrix} A_{pp} & & \\ & * & A_{qr} \\ & A_{rq} & A_{rr} \end{bmatrix}.$$

- ▶ DOFs p have been eliminated
- ▶ Interactions involving r are unchanged

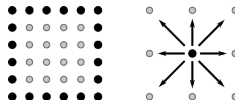
Interpolative decomposition

If $A_{:,q}$ has numerical rank k , then there exist

- ▶ **skeleton** (\hat{q}) and **redundant** (\check{q}) columns partitioning $q = \hat{q} \cup \check{q}$ with $|\hat{q}| = k$
- ▶ an interpolation matrix T_q

such that

$$A_{:, \check{q}} \approx A_{:, \hat{q}} T_q.$$



- ▶ Essentially a pivoted QR written slightly differently
- ▶ Rank-revealing to any specified precision $\epsilon > 0$

Interactions between separated regions are low-rank.

Skeletonization

- ▶ Efficient elimination of **redundant** DOFs
- ▶ Let $A = \begin{bmatrix} A_{pp} & A_{pq} \\ A_{qp} & A_{qq} \end{bmatrix}$ with A_{pq} and A_{qp} low-rank
- ▶ Apply ID to $\begin{bmatrix} A_{qp} \\ A_{pq}^* \end{bmatrix}$: $\begin{bmatrix} A_{q\check{p}} \\ A_{\check{p}q}^* \end{bmatrix} \approx \begin{bmatrix} A_{q\hat{p}} \\ A_{\hat{p}q}^* \end{bmatrix} T_p \implies \begin{aligned} A_{q\check{p}} &\approx A_{q\hat{p}} T_p \\ A_{\check{p}q} &\approx T_p^* A_{\hat{p}q} \end{aligned}$
- ▶ Reorder $A = \begin{bmatrix} A_{\check{p}\check{p}} & A_{\check{p}\hat{p}} & A_{\check{p}q} \\ A_{\hat{p}\check{p}} & A_{\hat{p}\hat{p}} & A_{\hat{p}q} \\ A_{q\check{p}} & A_{q\hat{p}} & A_{qq} \end{bmatrix}$, define $Q_p = \begin{bmatrix} I & & \\ -T_p & I & \\ & & I \end{bmatrix}$
- ▶ Sparsify via ID: $Q_p^* A Q_p \approx \begin{bmatrix} * & * & \\ * & A_{\hat{p}\hat{p}} & A_{\hat{p}q} \\ & A_{q\hat{p}} & A_{qq} \end{bmatrix} \xrightarrow{\text{elim}} \begin{bmatrix} * & & \\ & * & A_{\hat{p}q} \\ & A_{q\hat{p}} & A_{qq} \end{bmatrix}$
- ▶ Reduces to a subsystem involving **skeletons** only

Algorithm: recursive skeletonization factorization

Build quadtree/octree.

for each level $\ell = 0, 1, 2, \dots, L$ from finest to coarsest **do**

Let C_ℓ be the set of all cells on level ℓ .

for each cell $c \in C_\ell$ **do**

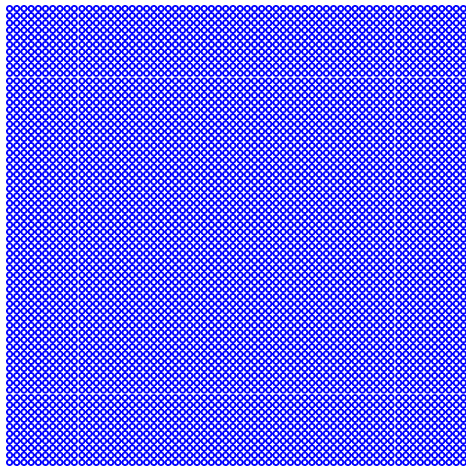
Skeletonize remaining DOFs in c .

end for

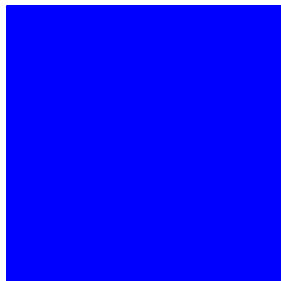
end for

- Reformulation of old algorithm using new elimination framework

RSF in 2D: level 0

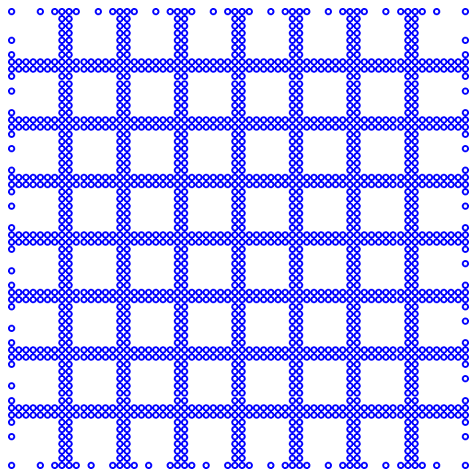


domain

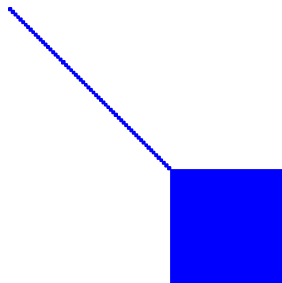


matrix

RSF in 2D: level 1

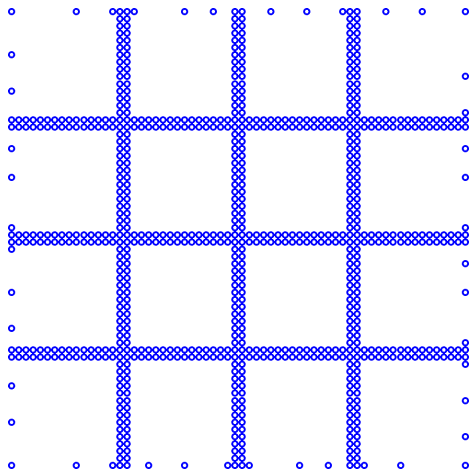


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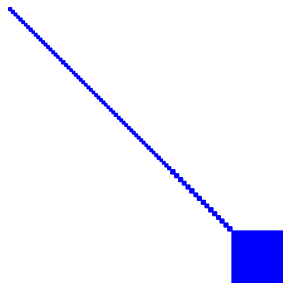


matrix

RSF in 2D: level 2

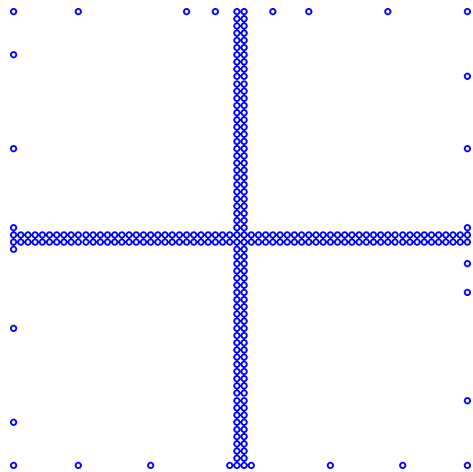


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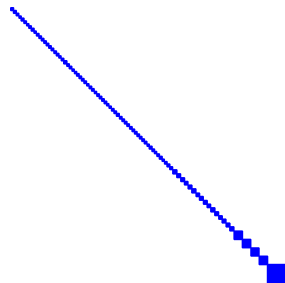


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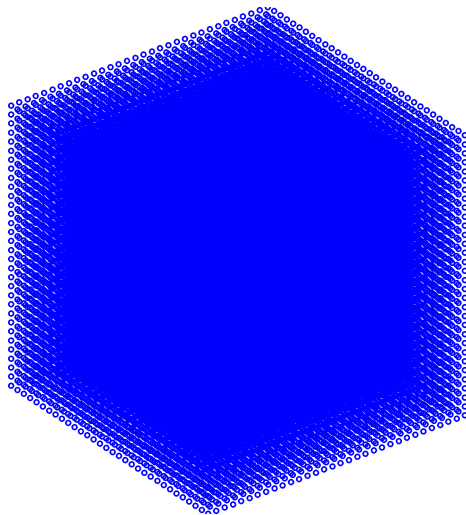
RSF in 2D: level 3



domain

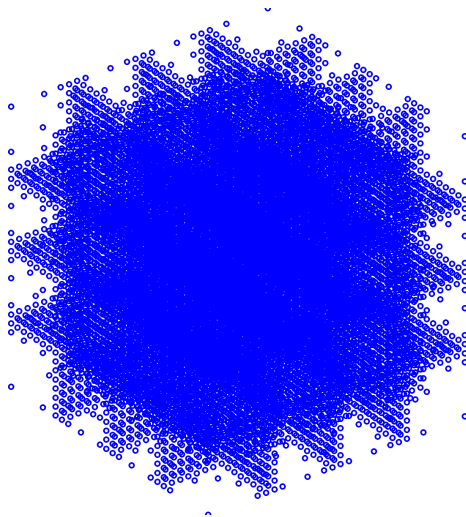


matrix



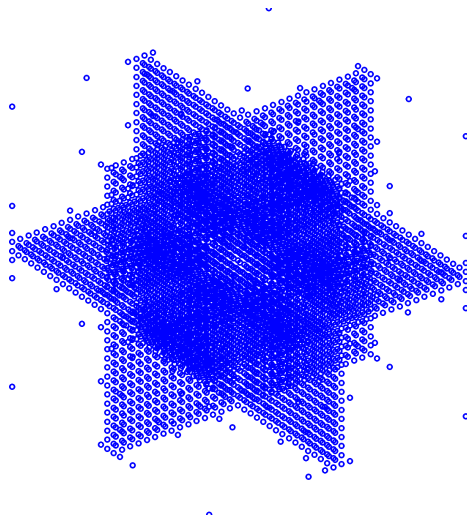
domain

RSF in 3D: level 1



domain

RSF in 3D: level 2



domain

- Skeletonization operators:

$$U_\ell = \prod_{c \in C_\ell} Q_c R_{\check{c}}, \quad V_\ell = \prod_{c \in C_\ell} Q_c S_{\check{c}}$$

$$Q_c = \begin{bmatrix} I & & \\ * & I & \\ & & I \end{bmatrix}, \quad R_{\check{c}}, S_{\check{c}} = \begin{bmatrix} I & * & \\ & I & \\ & & I \end{bmatrix}$$

- Block diagonalization:

$$D \approx U_{L-1}^* \cdots U_0^* A V_0 \cdots V_{L-1}$$

- Generalized LU decomposition:

$$A \approx U_0^{-*} \cdots U_{L-1}^{-*} D V_{L-1}^{-1} \cdots V_0^{-1}$$

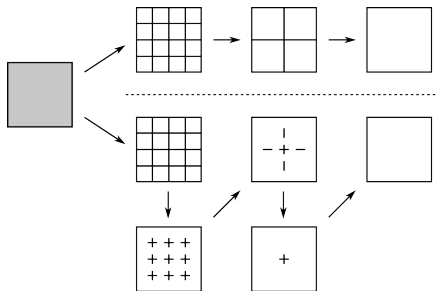
$$A^{-1} \approx V_0 \cdots V_{L-1} D^{-1} U_L^* \cdots U_0^*$$

- Fast direct **solver** or preconditioner

The cost is determined by the skeleton size.

	1D	2D	3D
Skeleton size	$O(\log N)$	$O(N^{1/2})$	$O(N^{2/3})$
Factorization cost	$O(N)$	$O(N^{3/2})$	$O(N^2)$
Solve cost	$O(N)$	$O(N \log N)$	$O(N^{4/3})$

Question: How to reduce the skeleton size in 2D and 3D?



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- ▶ Skeletons cluster near cell interfaces (Green's theorem)
- ▶ Exploit skeleton geometry by further skeletonizing **along interfaces**
- ▶ Dimensional reduction

Algorithm: hierarchical interpolative factorization for IEs in 2D

Build quadtree.

for each level $\ell = 0, 1, 2, \dots, L$ from finest to coarsest **do**

Let C_ℓ be the set of all **cells** on level ℓ .

for each cell $c \in C_\ell$ **do**

Skeletonize remaining DOFs in c .

end for

Let $C_{\ell+1/2}$ be the set of all **edges** on level ℓ .

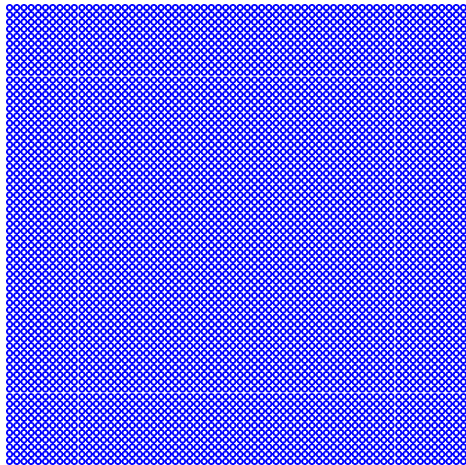
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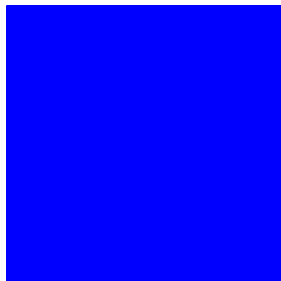
end for

end for

HIF-IE in 2D: level 0

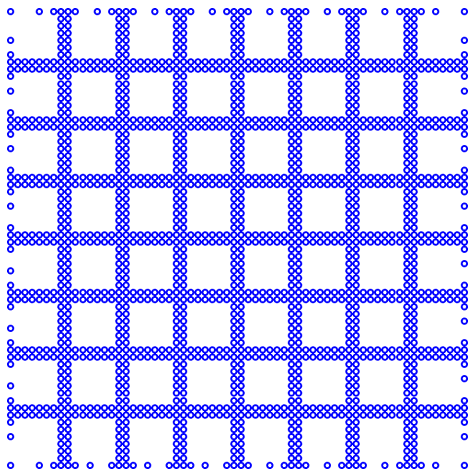


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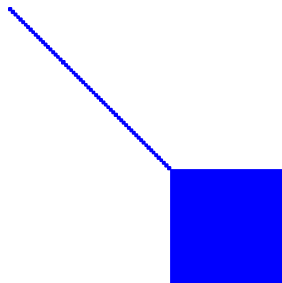


matrix

HIF-IE in 2D: level 1/2

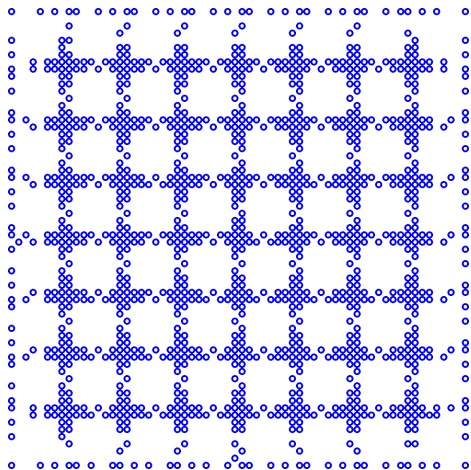


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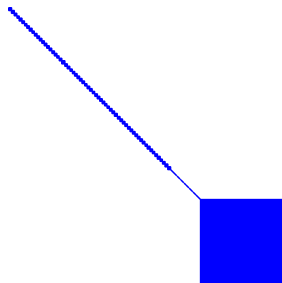


matrix

HIF-IE in 2D: level 1

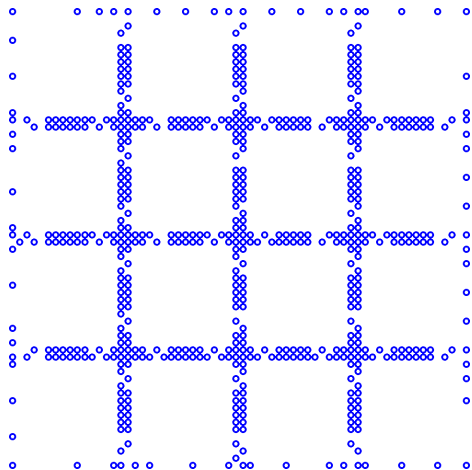


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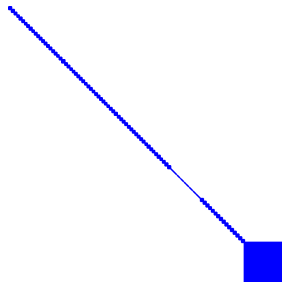


matrix

HIF-IE in 2D: level 3/2

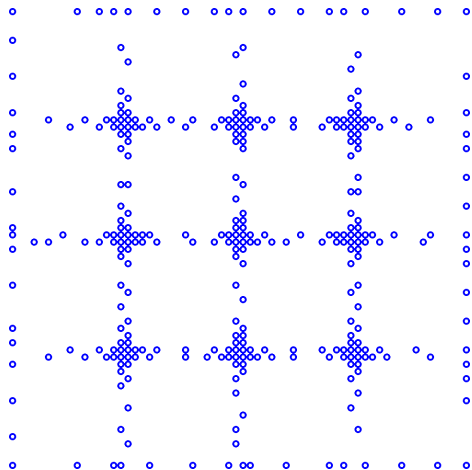


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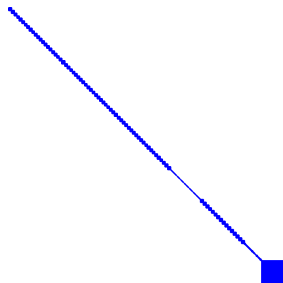


matrix

HIF-IE in 2D: level 2

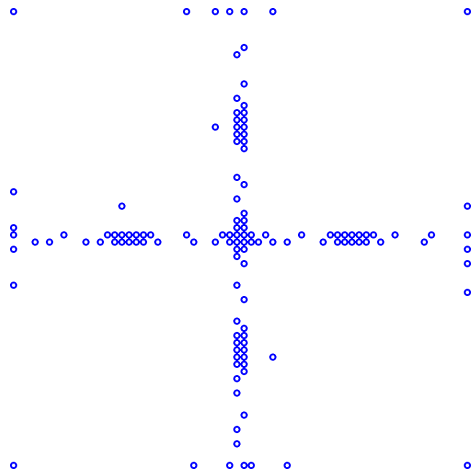


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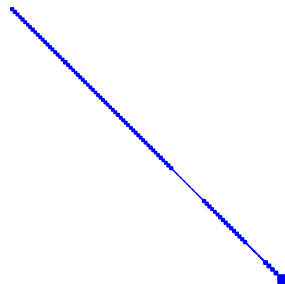


matrix

HIF-IE in 2D: level 5/2

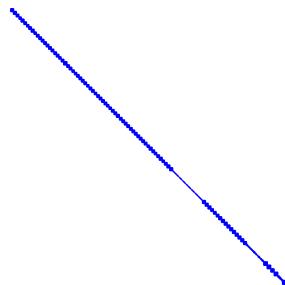
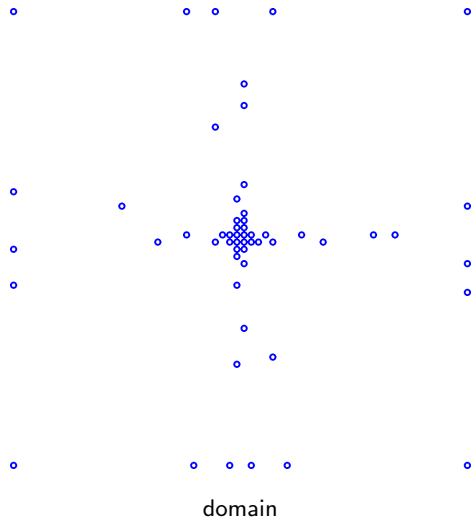


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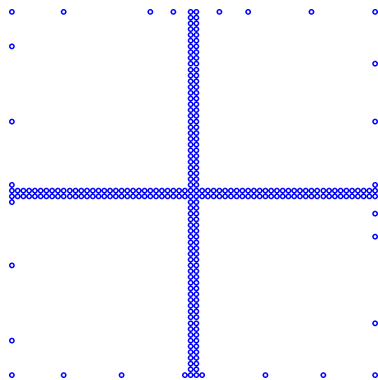


matrix

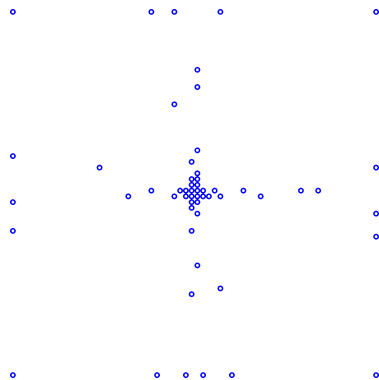
HIF-IE in 2D: level 3



RSF vs. HIF-IE in 2D

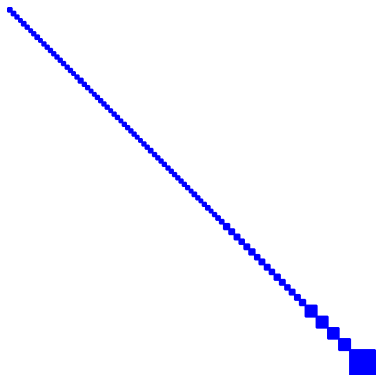


RSF

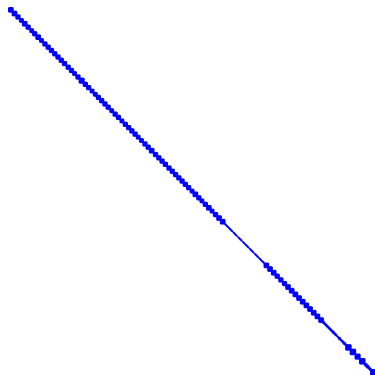


HIF-IE

RSF vs. HIF-IE in 2D



RSF



HIF-IE

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Let C_ℓ be the set of all **cells** on level ℓ .

for each cell $c \in C_\ell$ **do**

Skeletonize remaining DOFs in c .

end for

Let $C_{\ell+1/3}$ be the set of all **faces** on level ℓ .

for each cell $c \in C_{\ell+1/3}$ **do**

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Let $C_{\ell+2/3}$ be the set of all **edges** on level ℓ .

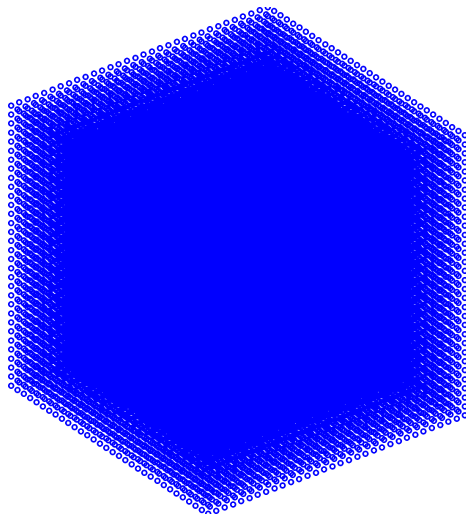
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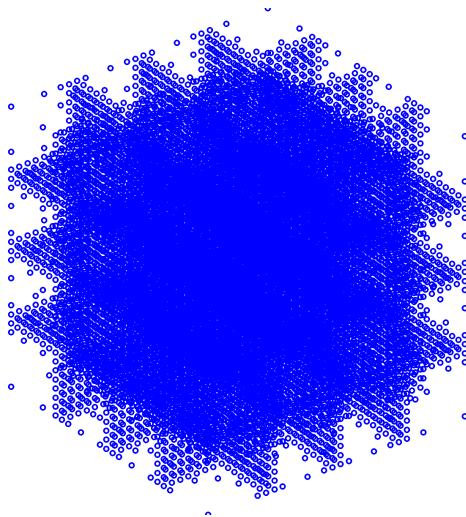
end for

HIF-IE in 3D: level 0

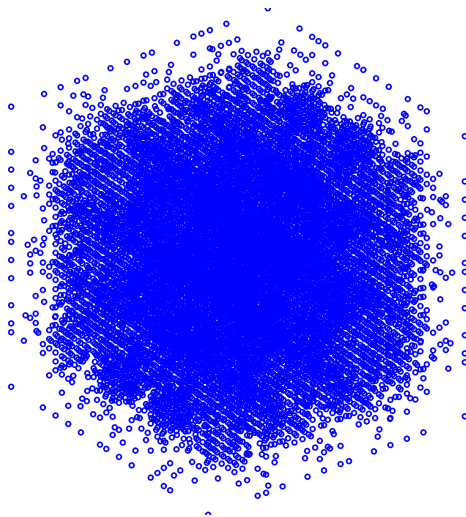


domain

HIF-IE in 3D: level 1/3

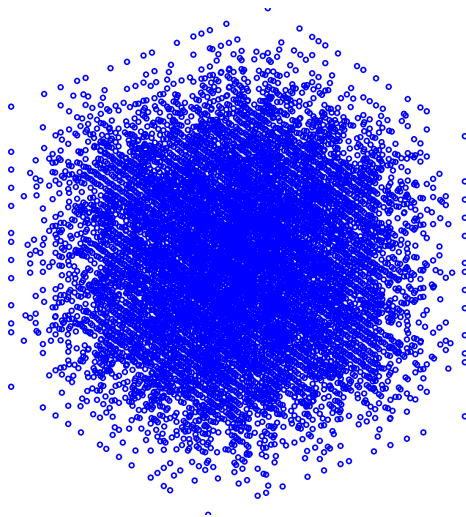


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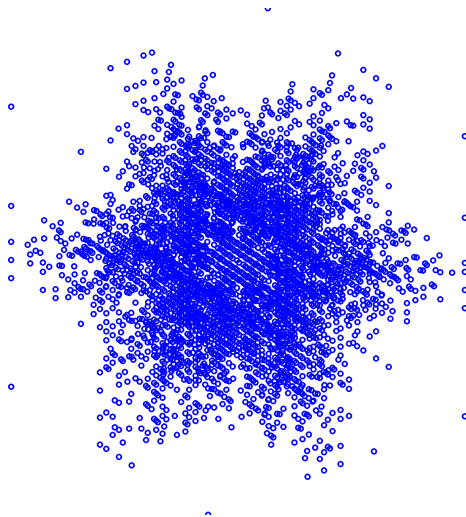


domain

HIF-IE in 3D: level 1

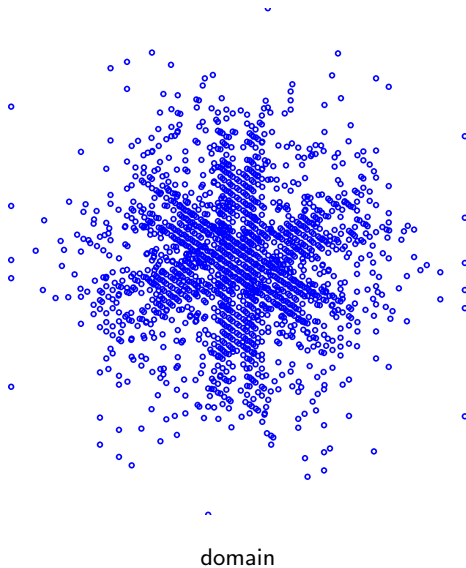


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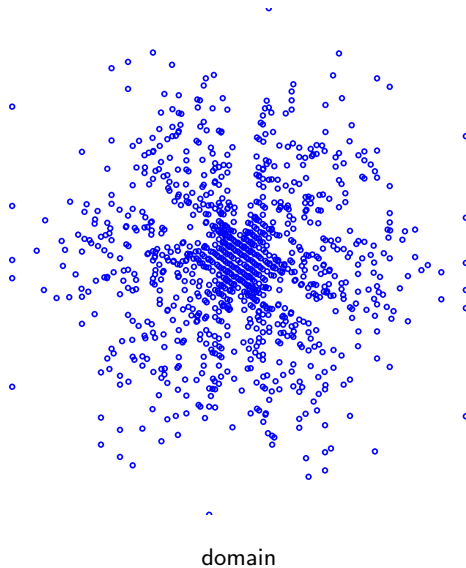


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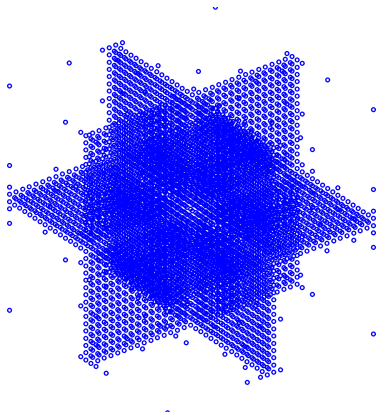
HIF-IE in 3D: level 5/3



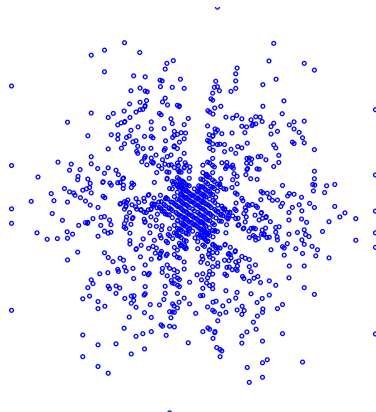
HIF-IE in 3D: level 2



RSF vs. HIF-IE in 3D



RSF



HIF-IE

► 2D:
$$A \approx U_0^{-*} U_{1/2}^{-*} \cdots U_{L-1/2}^{-*} D V_{L-1/2}^{-1} \cdots V_{1/2}^{-1} V_0^{-1}$$

$$A^{-1} \approx V_0 V_{1/2} \cdots V_{L-1/2} D^{-1} U_{L-1/2}^* \cdots U_{1/2}^* U_0^*$$

► 3D:
$$A \approx U_0^{-*} U_{1/3}^{-*} U_{2/3}^{-*} \cdots U_{L-1/3}^{-*} D V_{L-1/3}^{-1} \cdots V_{2/3}^{-1} V_{1/3}^{-1} V_0^{-1}$$

$$A^{-1} \approx V_0 V_{1/3} V_{2/3} \cdots V_{L-1/3} D^{-1} U_{L-1/3}^* \cdots U_{2/3}^* U_{1/3}^* U_0^*$$

Conjecture:

Skeleton size:	$O(\log N)$
Factorization cost:	$O(N)$
Solve cost:	$O(N)$

► 2D:
$$A \approx U_0^{-*} U_{1/2}^{-*} \cdots U_{L-1/2}^{-*} D V_{L-1/2}^{-1} \cdots V_{1/2}^{-1} V_0^{-1}$$

$$A^{-1} \approx V_0 V_{1/2} \cdots V_{L-1/2} D^{-1} U_{L-1/2}^* \cdots U_{1/2}^* U_0^*$$

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Conjecture:

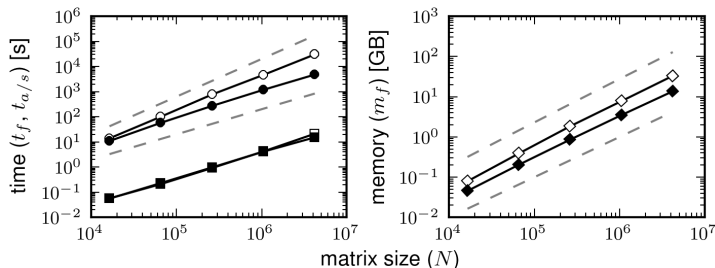
Skeleton size:	$O(\log N)$
Factorization cost:	$O(N)$
Solve cost:	$O(N)$

Actually slightly more complicated . . .

Numerical results in 2D

First-kind volume IE on the unit **square**:

$$-\frac{1}{2\pi} \int_{(0,1)^2} \log \|x - y\| u(y) dA(y) = f(x)$$

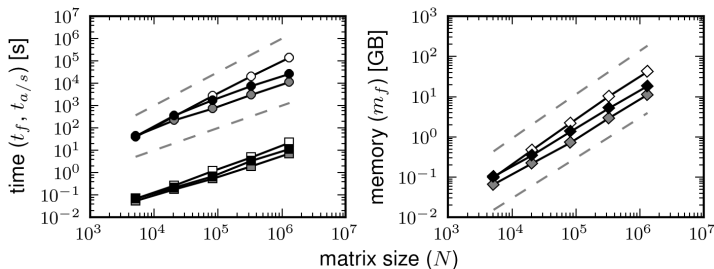


- ▶ **rsf2f2** (white), **hif2f2** (black)
- ▶ Factorization time (○), solve time (□), memory (◇) at precision $\epsilon = 10^{-6}$
- ▶ Reference scalings (gray dashes):
 - Left: $O(N)$ and $O(N^{3/2})$
 - Right: $O(N)$ and $O(N \log N)$

Numerical results in 3D

Second-kind boundary IE on the unit **sphere**:

$$-\frac{1}{2}u(x) + \frac{1}{4\pi} \int_{S^2} \frac{\partial}{\partial \nu(y)} \left(\frac{1}{\|x - y\|} \right) u(y) dS(y) = f(x)$$

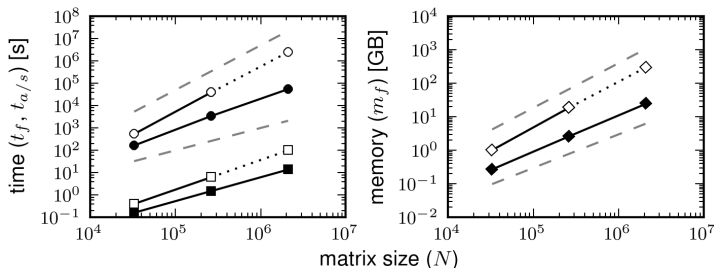


- ▶ **rskelf3** (white), **hifie3** (gray), **hifie3x** (black)
- ▶ Factorization time (○), solve time (□), memory (◇) at precision $\epsilon = 10^{-3}$
- ▶ Reference scalings (gray dashes):
 - Left: $O(N)$ and $O(N^{3/2})$
 - Right: $O(N)$ and $O(N \log N)$

Numerical results in 3D

First-kind volume IE on the unit **cube**:

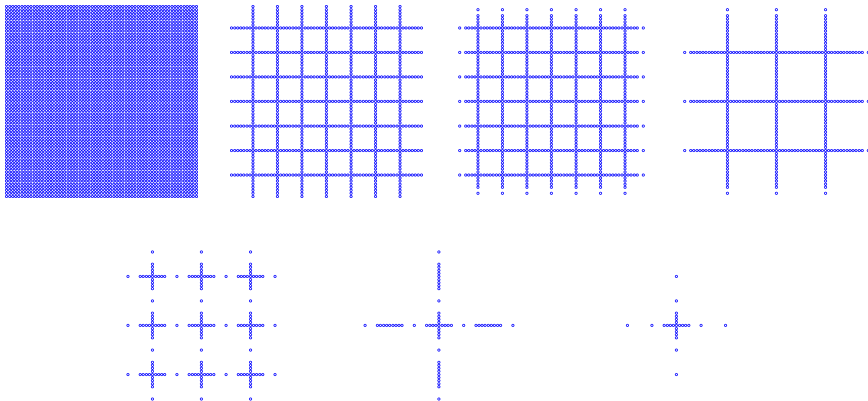
$$\frac{1}{4\pi} \int_{(0,1)^3} \frac{u(y)}{\|x - y\|} dV(y) = f(x)$$



- ▶ **rskelf3** (white), **hifie3** (black)
- ▶ Factorization time (○), solve time (□), memory (◇) at precision $\epsilon = 10^{-3}$
- ▶ Reference scalings (gray dashes):
 - Left: $O(N)$ and $O(N^2)$
 - Right: $O(N)$ and $O(N^{4/3})$

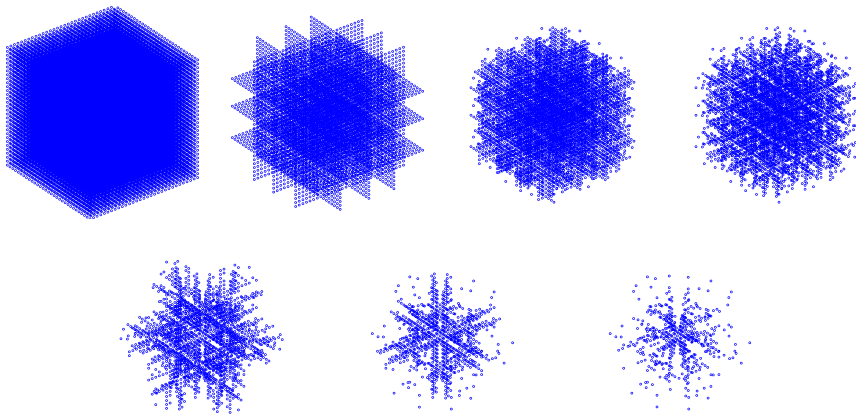
Hierarchical interpolative factorization for PDEs in 2D

- Build on top of multifrontal to exploit **sparsity**



Hierarchical interpolative factorization for PDEs in 3D

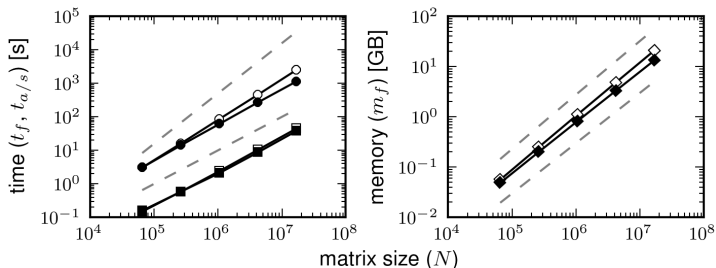
- Build on top of multifrontal to exploit **sparsity**



Numerical results in 2D

Five-point stencil on the unit **square** with $a(x) = 1$:

$$-\nabla \cdot (a(x) \nabla u(x)) = f(x)$$

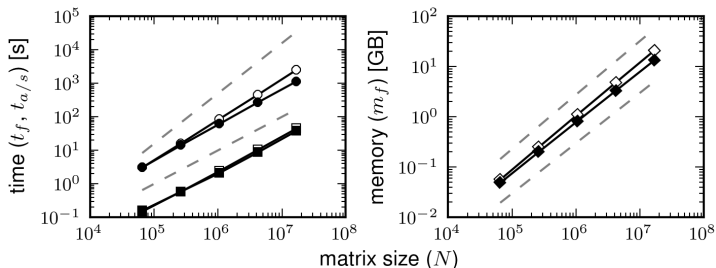


- ▶ **mf2** (white), **hifde2** (black)
- ▶ Factorization time (\circ), solve time (\square), memory (\diamond) at precision $\epsilon = 10^{-9}$
- ▶ Reference scalings (gray dashes):
 - Left: $O(N)$ and $O(N^{3/2})$
 - Right: $O(N)$ and $O(N \log N)$

Numerical results in 2D

Five-point stencil on the unit square with $a(x)$ a quantized high-contrast ($\kappa \sim 10^4$) random field:

$$-\nabla \cdot (a(x) \nabla u(x)) = f(x)$$



- ▶ **mf2** (white), **hifde2** (black)
- ▶ Factorization time (\circ), solve time (\square), memory (\diamond) at precision $\epsilon = 10^{-9}$
- ▶ Reference scalings (gray dashes):
 - Left: $O(N)$ and $O(N^{3/2})$
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Remarks on HIF

- ▶ Efficient **factorization** of structured operators in 2D/3D
- ▶ Empirical linear complexity but no proof yet
- ▶ Approximate generalized LU decomposition
 - Fast direct solver or preconditioner
 - Extremely effective for multiple RHS's
- ▶ **Extensions:** $A^{1/2}$, $\det A$, $\text{diag } A^{-1}$
- ▶ Highly parallelizable [with A. Benson, Y. Li, J. Poulson, L. Ying]
- ▶ MATLAB codes available at <https://github.com/klho/FLAM/>

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- ▶ *Perspective:* structured dense matrices can be **sparsified** very efficiently
- ▶ Can borrow directly from sparse algorithms, e.g., RSF = MF
- ▶ What other features of sparse matrices can be exploited?

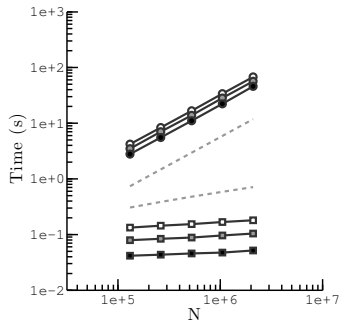
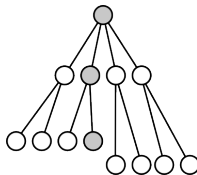
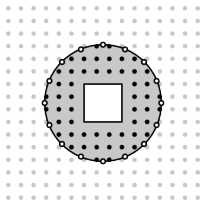
Local updating

“Naive” approach: local geometric perturbations as low-rank updates

- ▶ Sherman-Morrison-Woodbury: rank $k \implies O(Nk)$ cost
- ▶ Cannot accumulate updates across domain

Factorization updating

- ▶ Use Green's theorem to localize effect of perturbation
- ▶ Redo computation up only one branch of tree: $O(\log^\alpha N)$ cost



Conclusion

- ▶ Main thrust of my work: building technology for **structured matrices**
- ▶ Fast multiplication, direct solvers, least squares, factorizations
- ▶ Supporting tools: e.g., local updating
- ▶ **Outlook**: almost enough technology to make a deep run at some hard problems

Conclusion

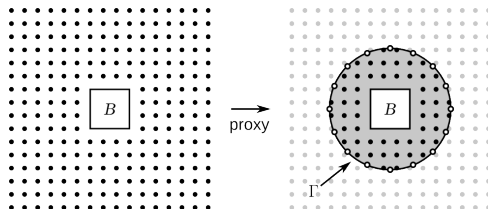
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- ▶ Fast multiplication, direct solvers, least squares, factorizations
- ▶ Supporting tools: e.g., local updating
- ▶ **Outlook:** almost enough technology to make a deep run at some hard problems
- ▶ Other related work:
 - Skeletonization/elimination as adaptive numerical coarsening
 - Butterfly algorithms for oscillatory kernels [with Y. Li, H. Yang, E. Martin, L. Ying]
- ▶ Next steps:
 - **Global** updating, spectral decompositions, matrix functions
 - Applications: biology, materials science, machine learning, UQ

References

- ▶ L. Greengard, K.L. Ho, J.-Y. Lee. A fast direct solver for scattering from periodic structures with multiple material interfaces in two dimensions. *J. Comput. Phys.* 258: 738–751, 2014.
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- ▶ K.L. Ho, L. Greengard. A fast direct solver for structured linear systems by recursive skeletonization. *SIAM J. Sci. Comput.* 34 (5): A2507–A2532, 2012.
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- ▶ K.L. Ho, L. Ying. Hierarchical interpolative factorization for elliptic operators: differential equations. Preprint, arXiv:1307.2895 [math.NA], 2013.
- ▶ K.L. Ho, L. Ying. Hierarchical interpolative factorization for elliptic operators: integral equations. Preprint, arXiv:1307.2666 [math.NA], 2013.
- ▶ V. Minden, A. Damle, K.L. Ho, L. Ying. A technique for updating hierarchical factorizations of integral operators. Preprint, arXiv:1411.5706 [math.NA], 2014.

Proxy compression

- ▶ Main cost of algorithm: computing IDs of tall-and-skinny matrices
- ▶ **Global** operation can be reduced to **local** operation using Green's theorem
- ▶ Suffices to compress against neighbors plus “proxy” surface
- ▶ Essential for overcoming $O(N^2)$ complexity



Second-kind IEs

- ▶ IEs of the form $a(x)u(x) + \int_{\Omega} K(x, y)u(y) d\Omega(y) = f(x)$
- ▶ **High contrast** in diagonal vs. off-diagonal entries
- ▶ Mixing of cell, face, edge in HIF-IE leads to error
- ▶ Need to use effective precision $O(\epsilon/N)$
- ▶ Quasilinear complexity estimates:

	2D	3D
Factorization cost	$O(N \log N)$	$O(N \log^6 N)$
Solve cost	$O(N \log \log N)$	$O(N \log^2 N)$