Skeleton preconditioner for coarse-graining integral equations

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Motivation

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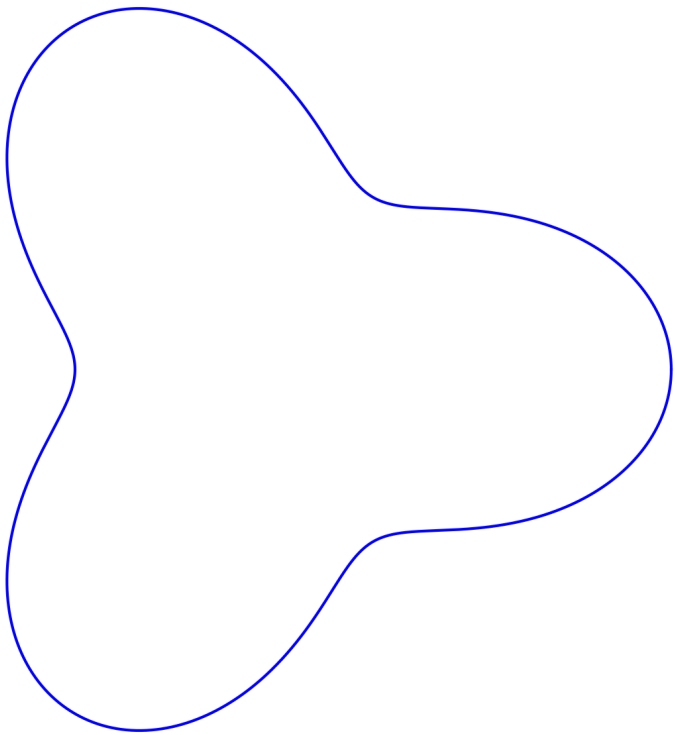

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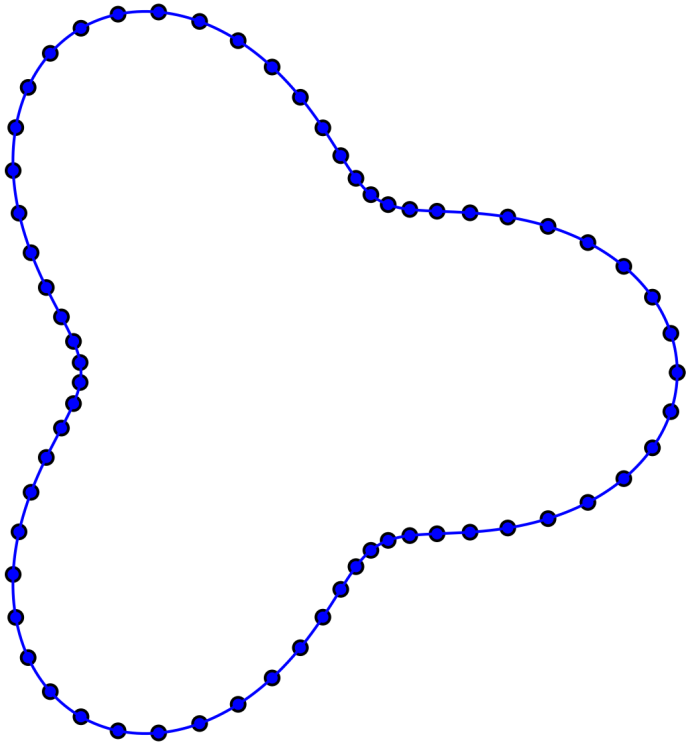

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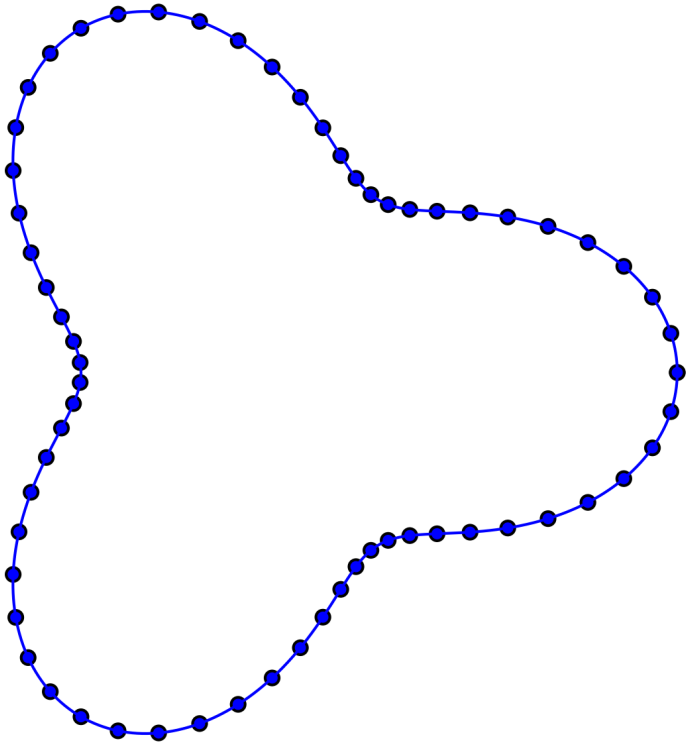

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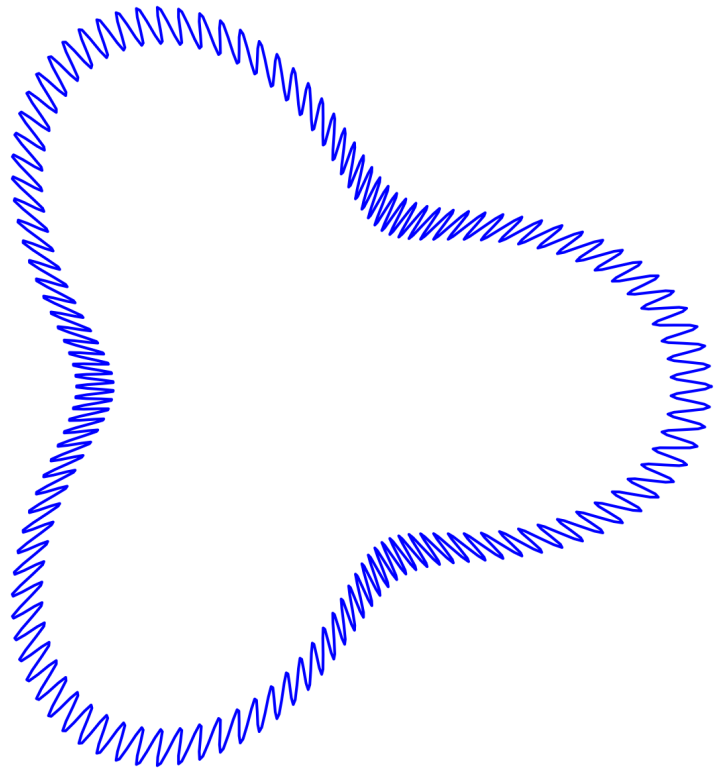

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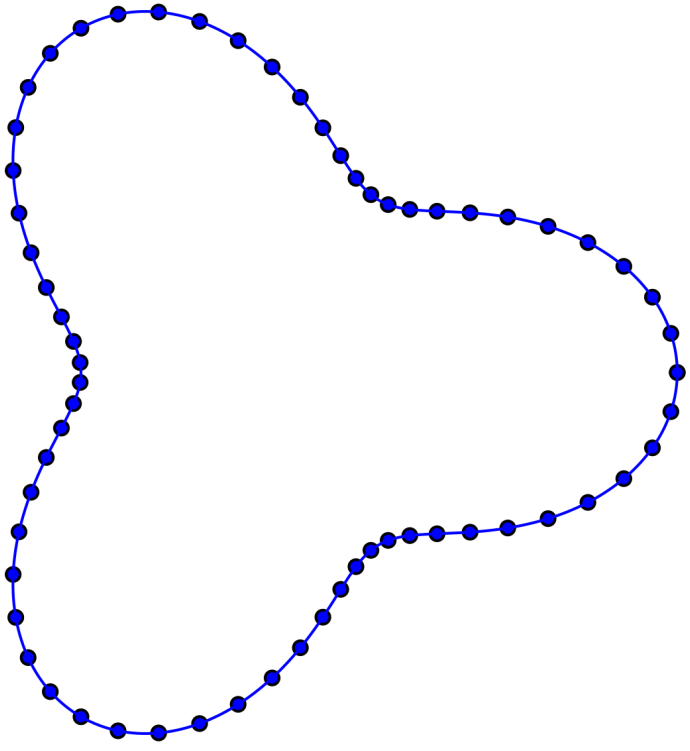

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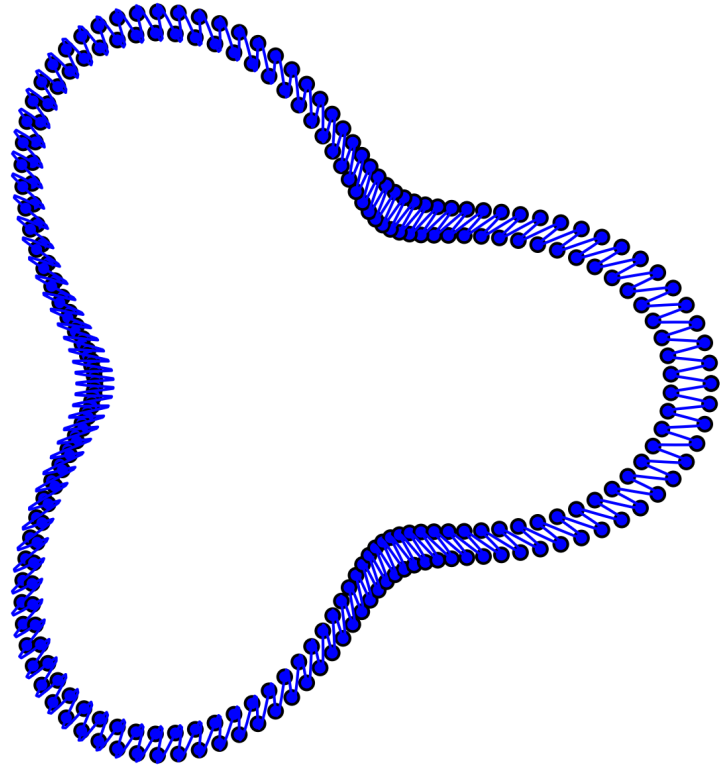

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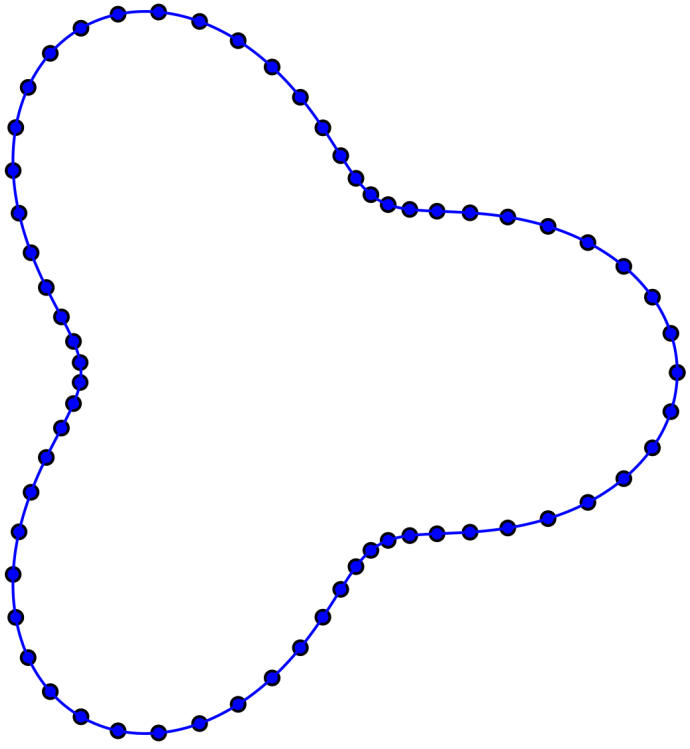

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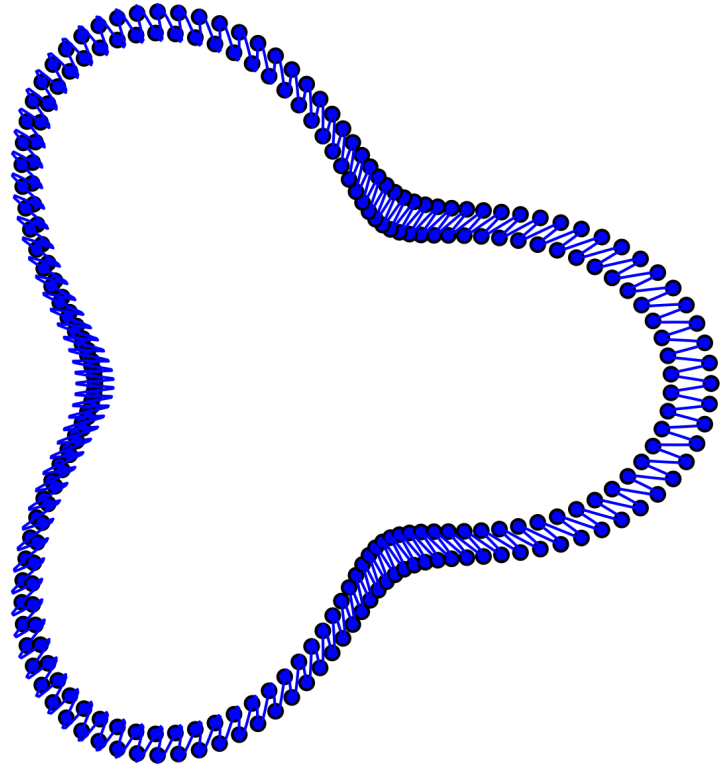

Forced to iterate on a system of high dimension...

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Can we do better? Inspired by RCIP (Helsing and Ojala), the answer is yes.

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- Fine-scale structure only needs to be seen locally
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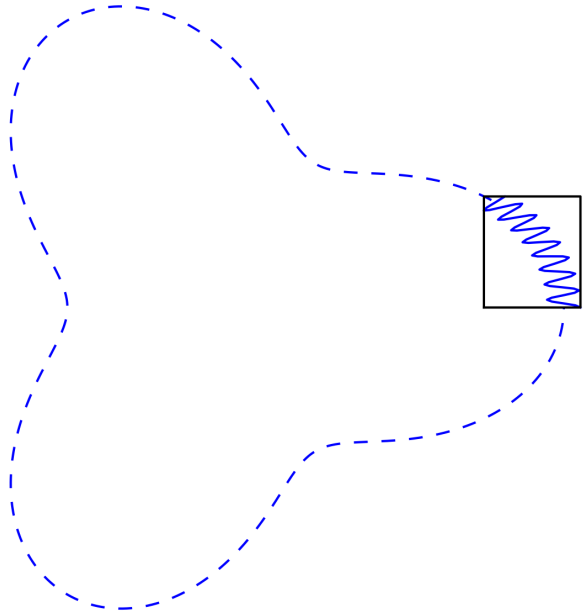
- Compute interactions between subdomains using skeletons
- Iterate on compressed skeleton system



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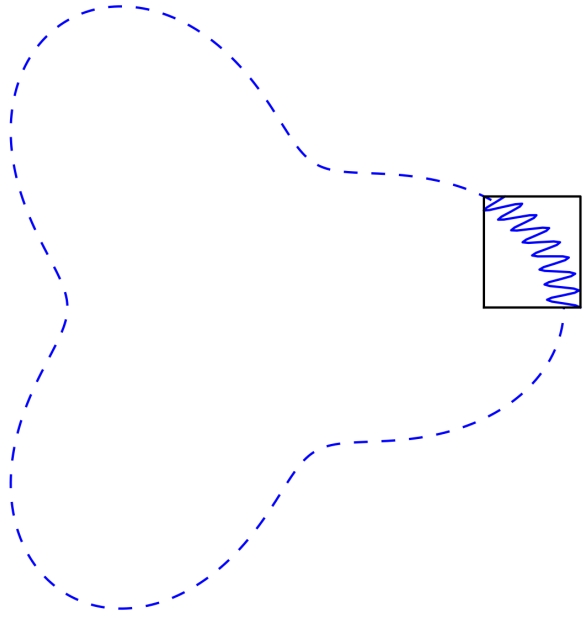


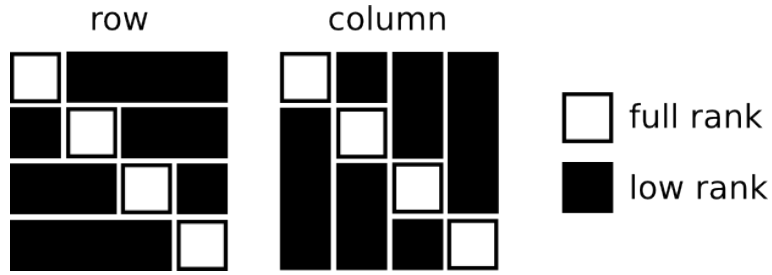
This talk in a nutshell: **RCIP in the language of skeletons**.

- All linear algebra; completely transparent; easy to apply, optimize, extend
- *Outline*: block separable matrix, interpolative decomposition, fast direct solver, skeleton preconditioner, numerical results

Block separable matrix

Low-rank off-diagonal block rows and columns:

$$\underbrace{\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}}_{A_{ij}} = \underbrace{\begin{bmatrix} \times \\ \times \end{bmatrix}}_{U_i} \underbrace{\begin{bmatrix} \times \end{bmatrix}}_{S_{ij}} \underbrace{\begin{bmatrix} \times & \times \end{bmatrix}}_{V_j^*}, \quad i \neq j$$



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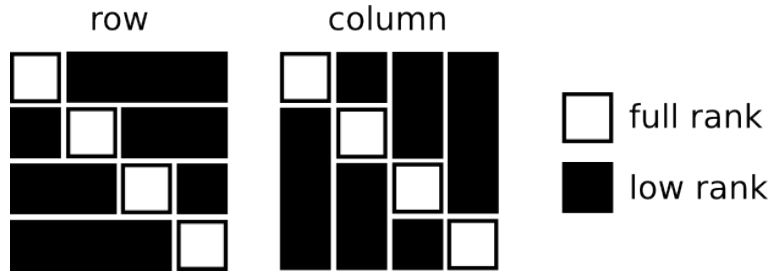
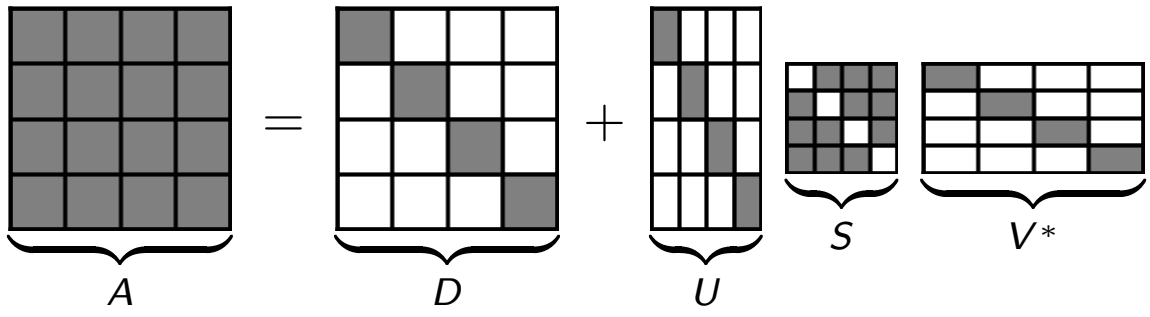


Matrix decomposition:



To solve $Ax = b$, let $z = V^*x$, $y = Sz$, $\Lambda = \left(V^*D^{-1}U\right)^{-1}$:

$$(\Lambda + S)z = \Lambda V^* D^{-1} b, \quad x = D^{-1}(b - Uy)$$

Interpolative decomposition

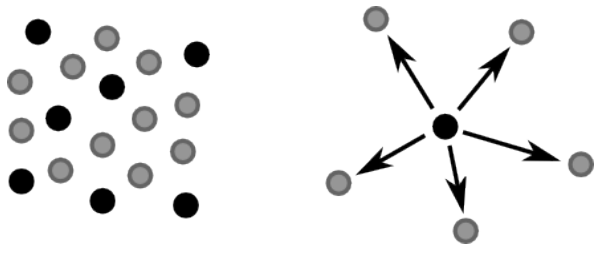
Factorization of low-rank matrix:

$$\underbrace{A}_{m\times n} = \underbrace{B}_{m\times k}\ \underbrace{P}_{k\times n}$$

- B is a column-submatrix of A
- P is an interpolation matrix (containing the $k \times k$ identity)

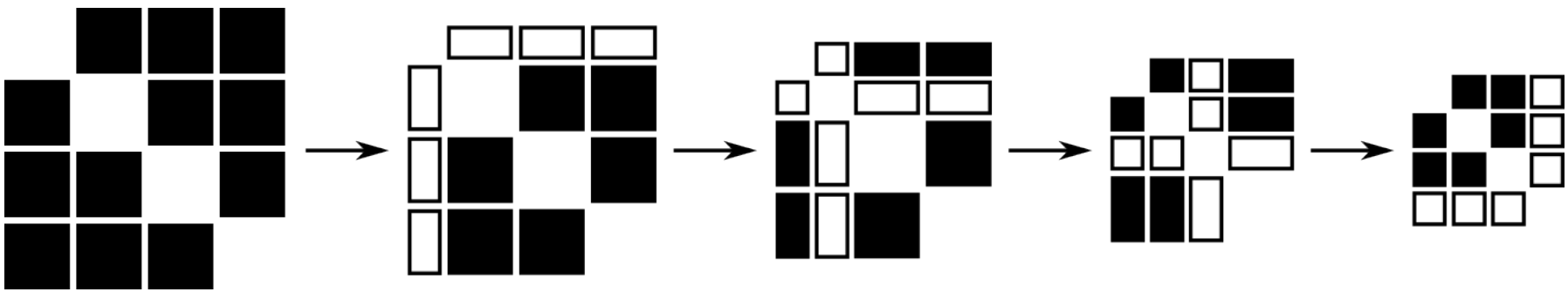
Essentially an RRQR written slightly differently

- ID compresses the column space; to compress the row space, apply to A^*
- Retained rows and columns: skeletons
- Can adaptively compute the ID to any specified precision



Matrix compression

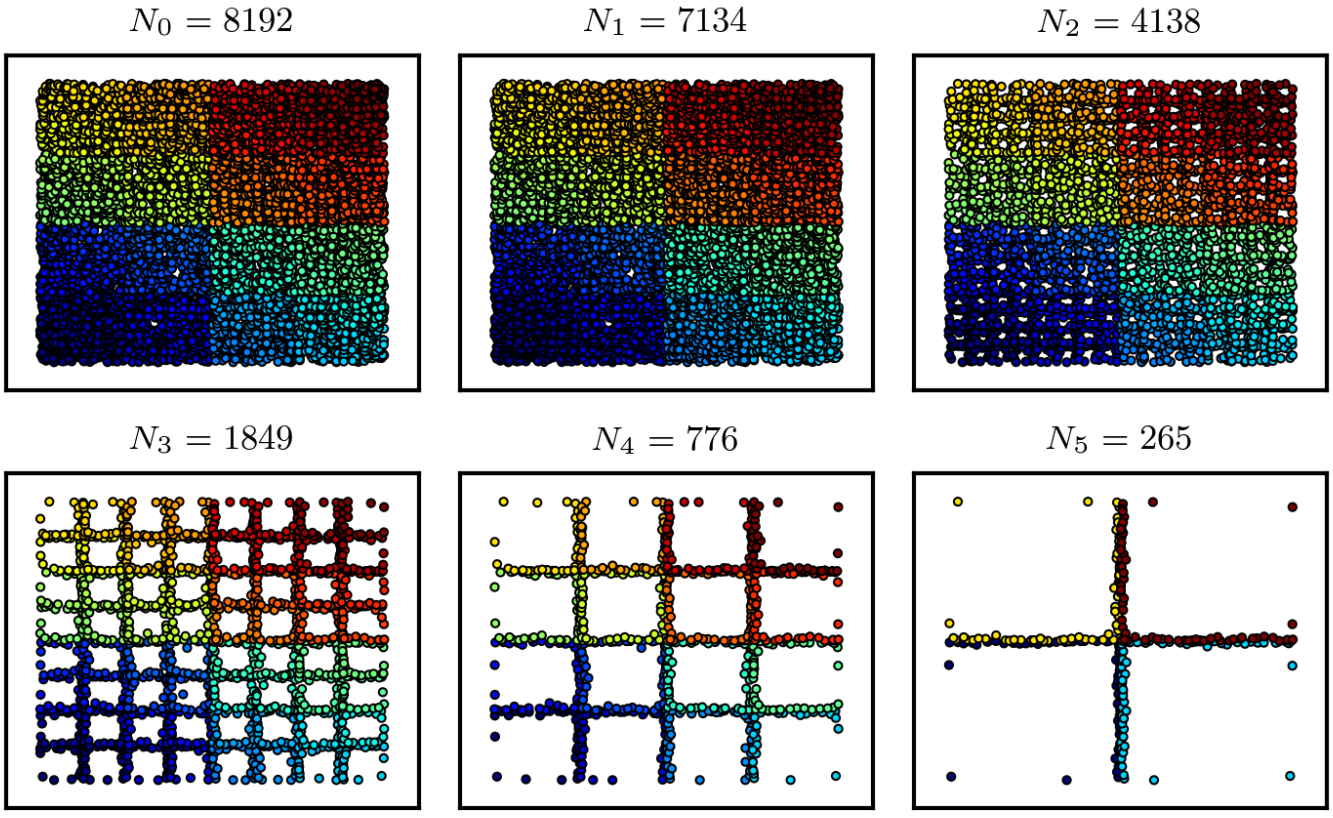
- Extract diagonal blocks D
- Compress row spaces of off-diagonal block rows $\longrightarrow U$
- Compress column spaces of off-diagonal block columns $\longrightarrow V^*$
- Subselect S from intersection of row and column skeletons
- Approximate off-diagonal blocks by $A_{ij} \approx U_i S_{ij} V_j^*$



Skeletonization

Fast direct solver

- Skeletonize matrix: $A = D + USV^*$
- Reduce $Ax = b$ to $(\Lambda + S)z = \Lambda V^* D^{-1} b$
- Since $\Lambda + S$ is "just like A", skeletonize again by working up a tree
- Repeat until at root, then solve densely



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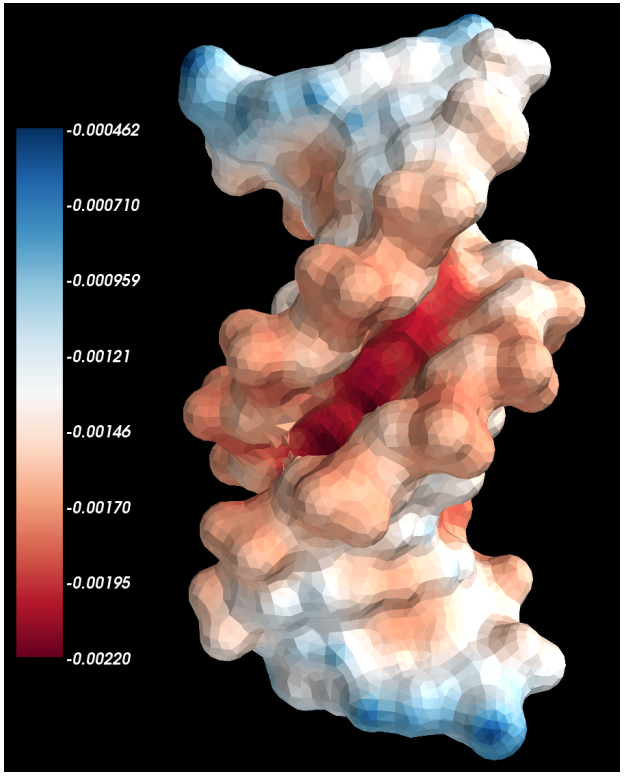

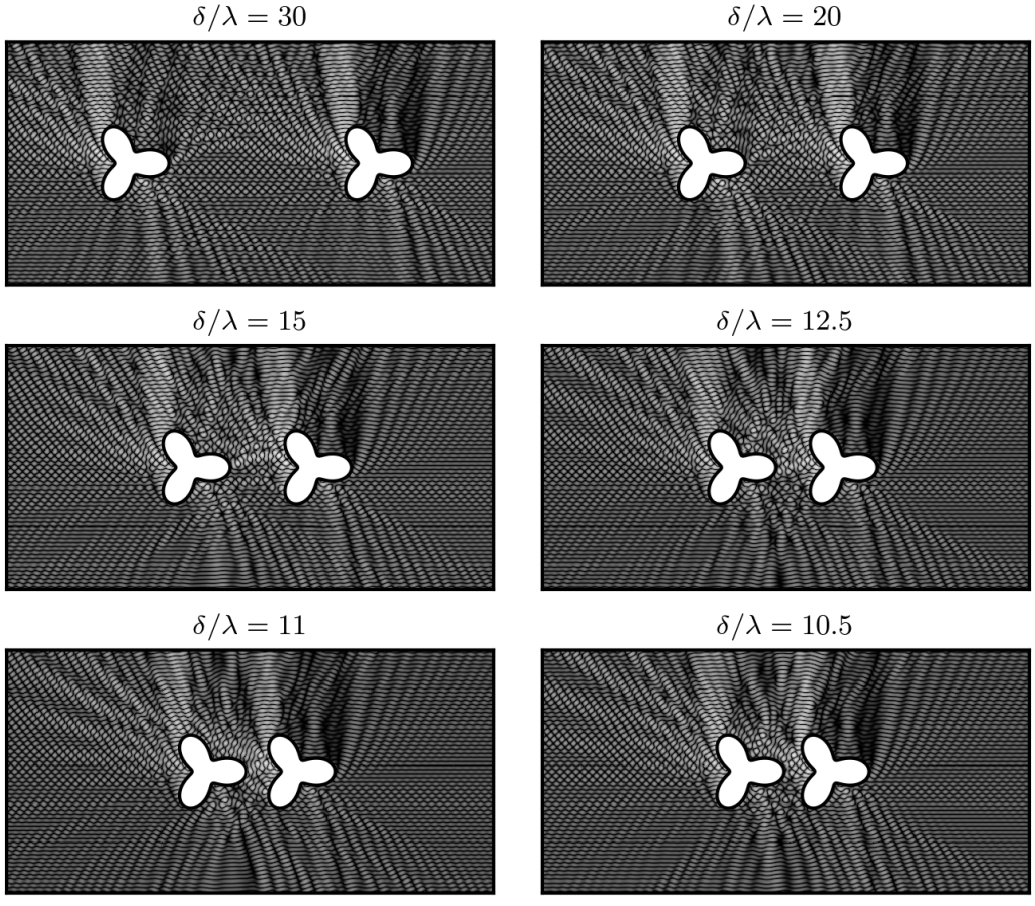

Remarks

- Many authors: Bremer, Gillman, Greengard, Martinsson, Rokhlin, ...
- Great for quasi-1D problems at low frequency (e.g., 2D Laplace BIE)
- Not so great in higher dimensions (yet) or at high frequency

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In many cases, iterating is still the best approach.

- But iterative solvers deal poorly with singularities
- Hybrid strategy: precondition with local direct solves
- Design goals:
  - Flexibility, generality, compatibility with fast algorithms
  - Iterate only on 'true' problem size

Skeleton preconditioner

- Block diagonal preconditioner:

$$D^{-1}Ax = D^{-1}b \quad \text{or} \quad (AD^{-1})Dx = b$$

  - Fast algorithms to invert D and apply A
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$$(\Lambda + S)z = \Lambda V^* D^{-1} b, \quad \Lambda = \left(V^* D^{-1} U\right)^{-1}$$

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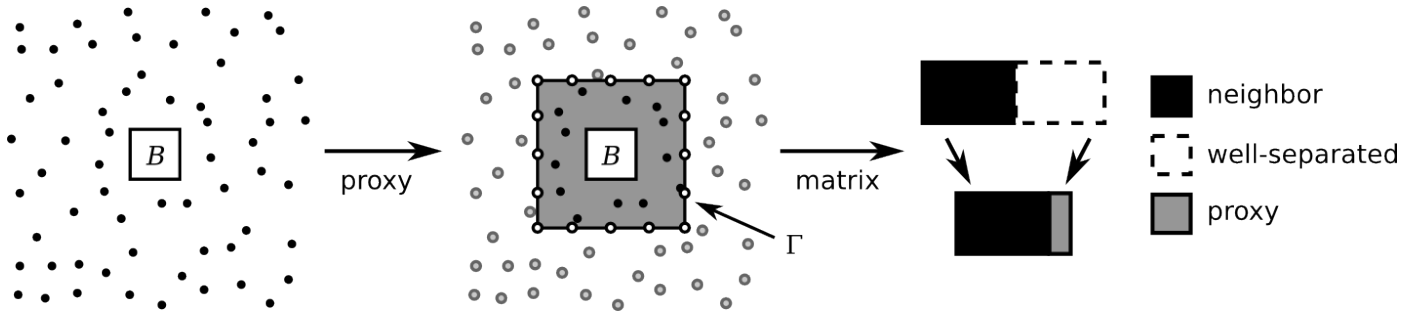
... and that's it!

Algorithm

- Choose subdomains (requires scale separation) and skeletonize
  - Based on feature size, wavelength, etc.
  - Compute IDs hierarchically and use proxy trick:



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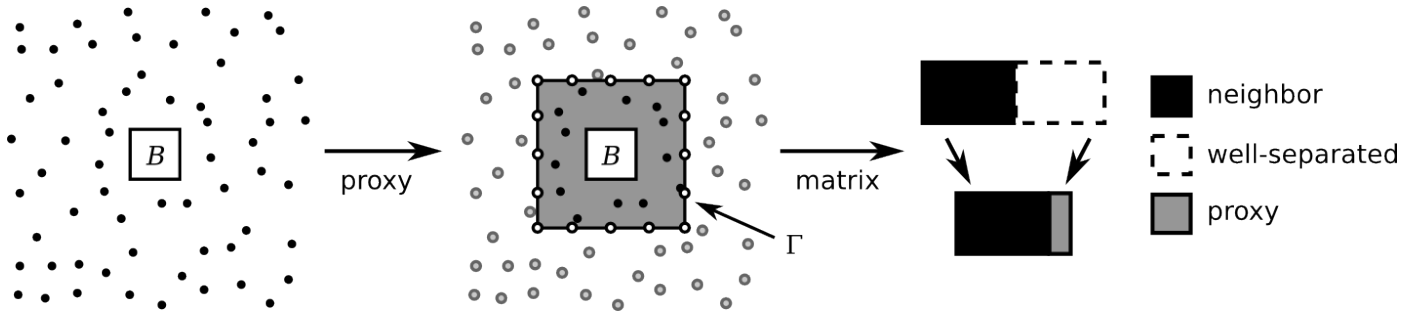


- Direct solve on each subdomain (fast direct solver, dense)
- Iterate on preconditioned skeleton system (fast multiplication)

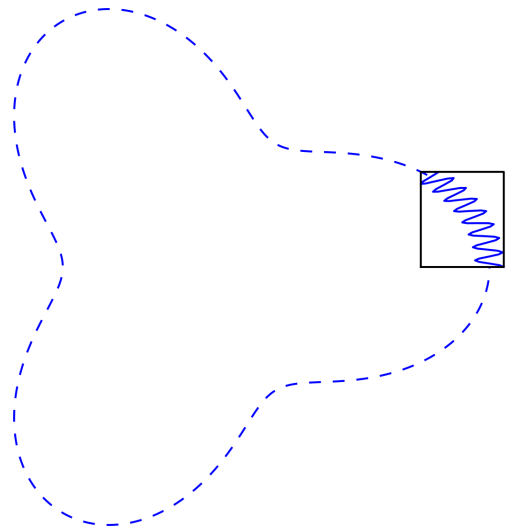
$$\left(I + V^* D^{-1} U S\right) z = V^* D^{-1} b$$

- Local solves to recover $x = D^{-1}(b - USz)$

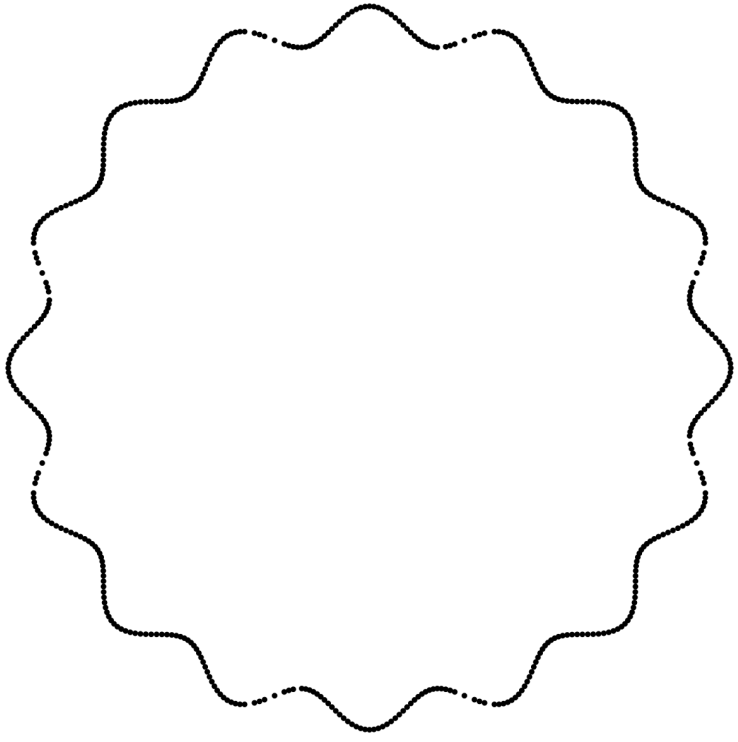
Remarks

- Coarse-grained description of subdomain interactions
  - x: fine variables
  - z: coarse variables, good outside of own subdomain
- Capture moments of solution (cf. moments of source in FMM)
- Evaluate field using coarse variables plus local 'corrections'
- If only far field needed (e.g., RCS), no further work!
- Explicit control over coarse basis during compression
- Example: Neumann problem
  - Single-layer representation: $u = \mathbb{S}\sigma$
  - Integral equation: $u' = \pm\frac{1}{2}\sigma + \mathbb{S}'\sigma$
  - Compress contributions from both $\mathbb{S}$ and $\mathbb{S}'$
- Same asymptotic complexity as iterative solver
- Just like RCIP but using

$$\left(I + V^* D^{-1} U S\right) z = V^* D^{-1} b$$



Numerical results



- Interior Dirichlet Helmholtz
- 16 bumps, 32–256 points each
- 8λ total, 2λ subdomains
- Tenth-order Kapur-Rokhlin
- Compress to $\epsilon = 10^{-14}$
- Dense linear algebra

p	N	K	$T^{\text{skel}}_{\text{comp}}$	$T^{\text{skel}}_{\text{dir}}$	$T^{\text{skel}}_{\text{iter}}$	$n^{\text{skel}}_{\text{iter}}$	$T^{\text{full}}_{\text{iter}}$	$n^{\text{full}}_{\text{iter}}$	error
32	512	484	0.24	0.01	0.32	155	1.0	373	8.4E−04
64	1024	859	0.80	0.03	0.80	138	4.1	475	3.9E−07
128	2048	977	1.86	0.10	0.91	107	5.2	221	3.0E−10
256	4096	757	4.24	0.90	1.61	101	13.5	145	7.3E−13

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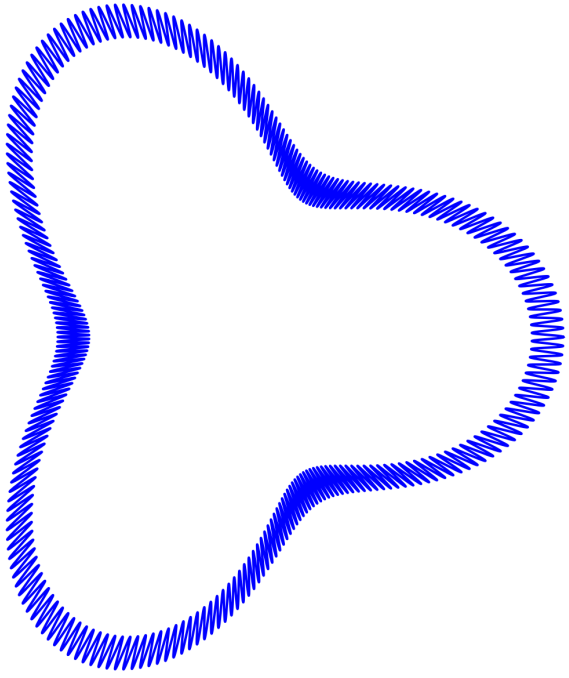

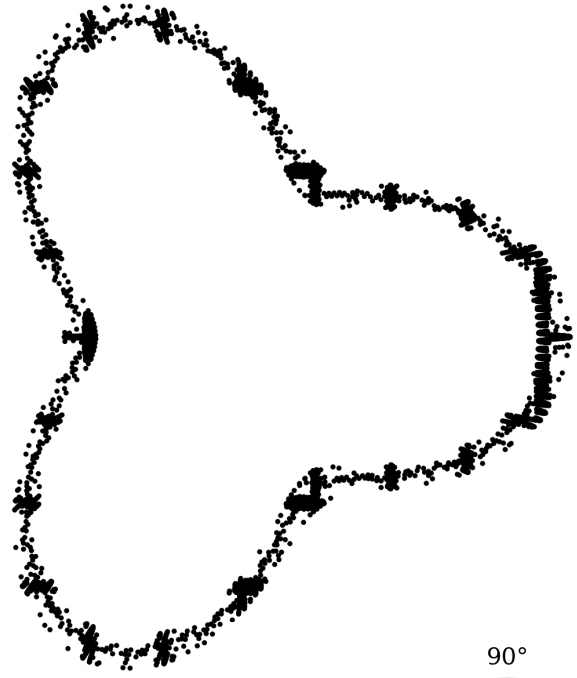

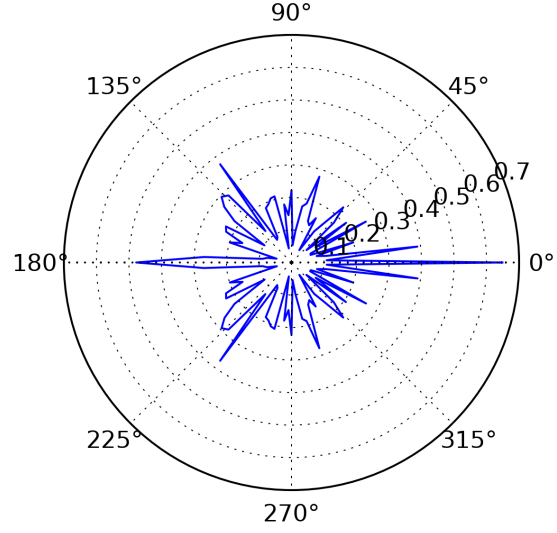

256 bumps, $\epsilon = 10^{-6}$

λ-size	p	N	K	T_{solv}	T_{rcs}
64 / 4	128	32768	4771	267	0.13
128 / 4	256	65536	11953	1419	0.34

Summary

- Skeleton preconditioner based on variant of fast direct solver reduction
- Related to RCIP (Helsing/Ojala) and charge bases (Bremer/Rokhlin/Sammis)
- Cleans up corners, boundary layers, sub-wavelength structures, etc.
- **Allows brute force discretization without much penalty**
- All overhead is local and parallelizable
- Compressed evaluation of far field
- Plug-and-play: fast direct solvers, fast multipole methods
- Very general: boundaries, volumes, whatever
- Formalizes philosophy of compress-then-iterate