Fibonacci numbers in nature

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What are the Fibonacci numbers?

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$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

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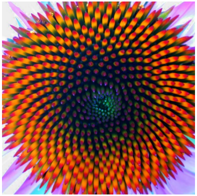
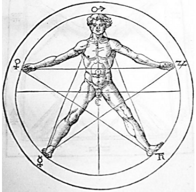
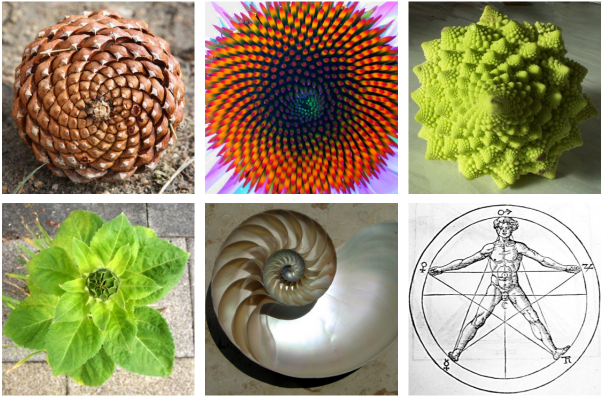






What are the Fibonacci numbers?

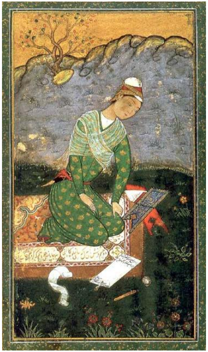
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .



One of these is not exactly related to the Fibonacci numbers.

A little history

- Studied in India as early as 200 BC



A little history

- Studied in India as early as 200 BC
- Introduced to the West by Leonardo of Pisa (Fibonacci) in *Liber Abaci* (1202)

Leonardo of Pisa

c. 1170 – c. 1250

A little history

- Studied in India as early as 200 BC
- Introduced to the West by Leonardo of Pisa (Fibonacci) in *Liber Abaci* (1202)
  - "Book of Calculation"
  - Described Hindu-Arabic numerals
  - Used Fibonacci numbers to model rabbit population growth

Leonardo of Pisa

c. 1170 – c. 1250

Bunnies!

Model assumptions

- One male-female pair originally
- Each pair able to mate at one month, mating each month thereafter
- Each mating produces one new pair after one month

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How many pairs are there after n months?

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1									

Month 1

- One pair originally

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	⋯	n
pairs	1	1								

Month 2

- From last month: 1
- Newly born: 0

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2							

Month 3

- From last month: 1
- Newly born: 1

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3						

Month 4

- From last month: 2
- Newly born: 1

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5					

Month 5

- From last month: 3
- Newly born: 2

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5	8				

Month 6

- From last month: 5
- Newly born: 3

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5	8	13			

Month 7

- From last month: 8
- Newly born: 5

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5	8	13	21		

Month 8

- From last month: 13
- Newly born: 8

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5	8	13	21	$\cdots$	F_n

Month n

- From last month: F_{n-1}
- Newly born: F_{n-2}

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
pairs	1	1	2	3	5	8	13	21	$\cdots$	F_n

$$F_1 = F_2 = 1 \qquad \text{(seed values)}$$

$$F_n = F_{n-1} + F_{n-2} \qquad \text{(recurrence relation)}$$

Fibonacci bunnies

month	1	2	3	4	5	6	7	8	$\cdots$	n
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$$F_0 = 0, \quad F_1 = 1 \qquad \text{(seed values)}$$

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Fibonacci bunnies

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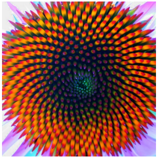
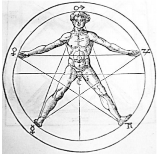
Note that the rabbit model is unrealistic (why?), but we will see a real instance where the Fibonacci numbers show up very shortly.

Fibonacci numbers in nature

n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
F_n	0	1	1	2	3	5	8	13	21	34	55	89	⋯





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Number of spirals

Clockwise:	13
Counterclockwise:	8

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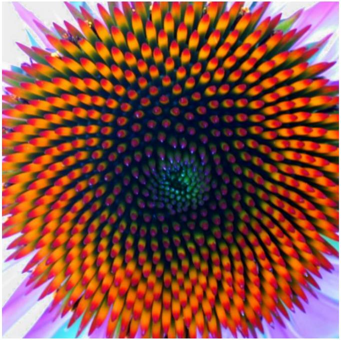

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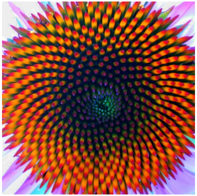



So where do the Fibonacci numbers come from?

A crash course on plant growth

- Central turning growing tip
- Emits new seed head, floret, leaf bud, etc. every α turns
- Seed heads grow outward with time

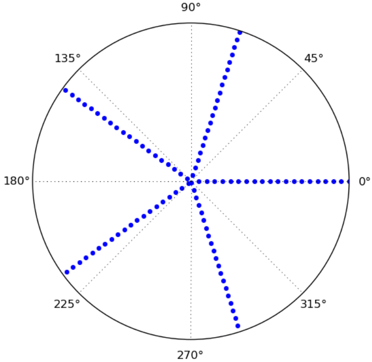
A crash course on plant growth

$\alpha = 1/4$

$\alpha = 1/5$

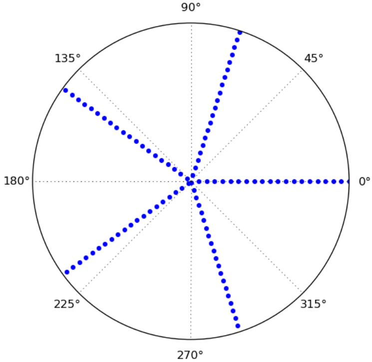
From a plant's perspective

- What's wrong with this growth pattern?



From a plant's perspective

- What's wrong with this growth pattern?
- Too much wasted space!



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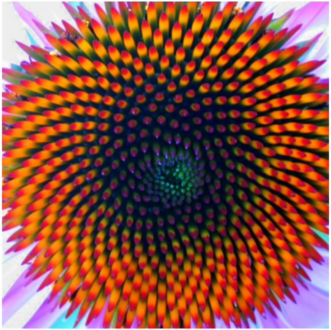

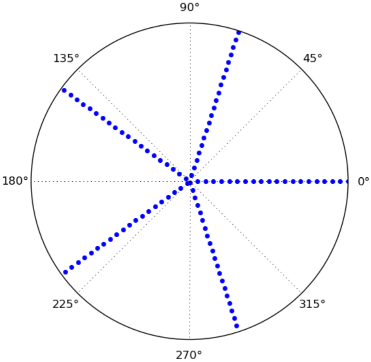

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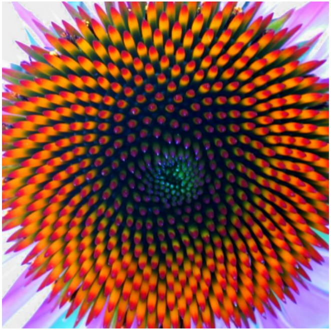
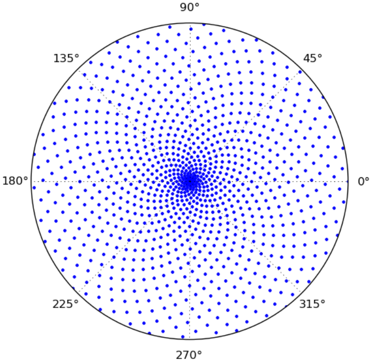
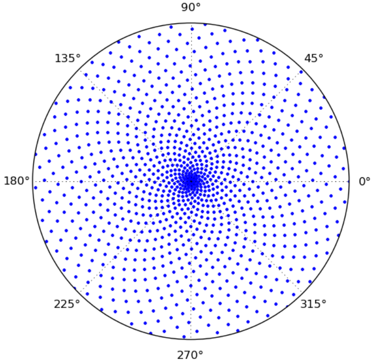




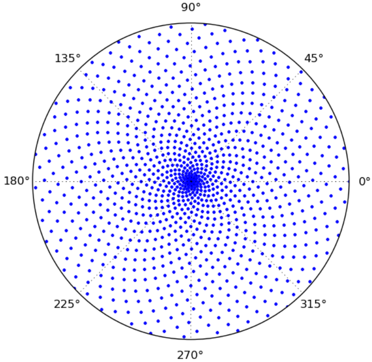
From a plant's perspective

- What's wrong with this growth pattern?
- Too much wasted space!
- Want to maximize exposure to sunlight, dew, CO_2



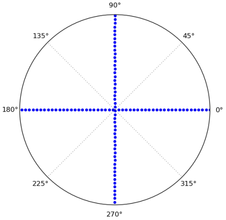
From a plant's perspective

- What's wrong with this growth pattern?
- Too much wasted space!
- Want to maximize exposure to sunlight, dew, CO_2
- Evolve for optimal packing

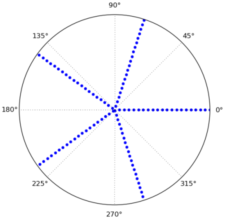


Floral showcase

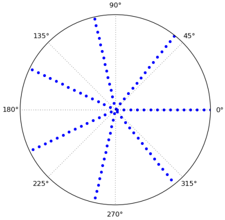
$\alpha = 1/4$



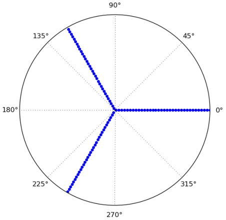
$\alpha = 1/5$



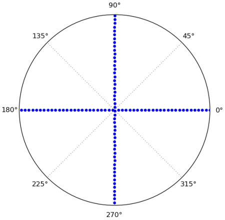
$\alpha = 1/7$



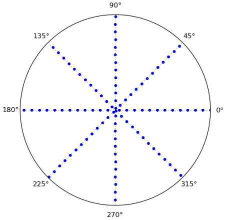
$\alpha = 2/3$



$\alpha = 3/4$



$\alpha = 5/8$



Rationality is not always good

Definition

A rational number is a number that can be expressed as a fraction m/n, where m and n are integers.

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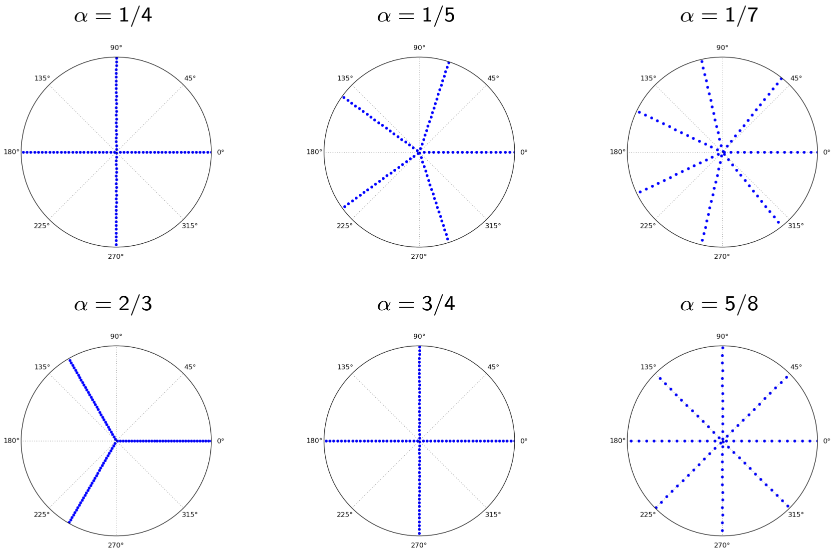
Can we get a good covering with $\alpha = m/n$? The answer is no.

Why?

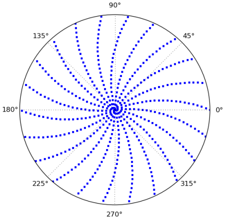
- Growing tip makes m revolutions every n seeds
- Growth pattern repeats after n seeds
- At most n "rays" of seeds

Floral showcase redux (rational)

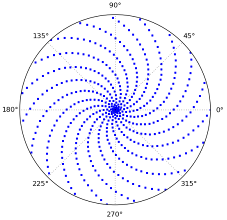


Floral showcase (irrational)

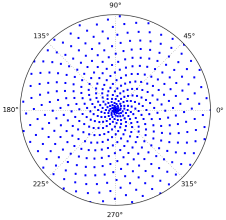
$\alpha = 1/\pi$



$\alpha = 1/e$



$\alpha = 1/\sqrt{2}$



Floral showcase (irrational)

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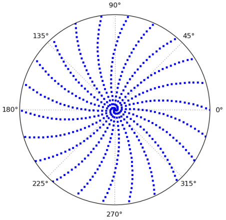
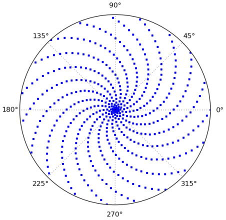


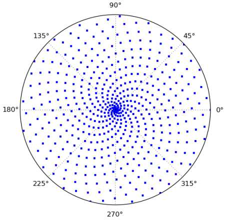
"Less" irrational

$\alpha = 1/e$



$\Longleftrightarrow$

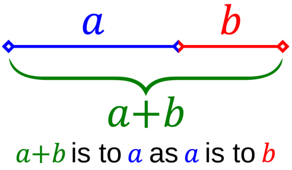
$\alpha = 1/\sqrt{2}$



"More" irrational

- Some irrationals work better than others.
- What is the "most" irrational number?

The golden ratio



Mathematically,

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi.$$

How to solve for φ?

The golden ratio

1. Given:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi$$

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2. Simplify: $$1 + \frac{b}{a} = \varphi$$

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5. Quadratic formula: $$\varphi = \frac{1 + \sqrt{5}}{2}$$

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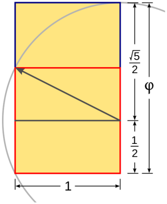
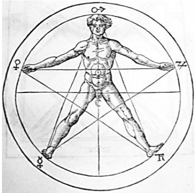
4. Rearrange: $$\varphi^2 - \varphi - 1 = 0$$
5. Quadratic formula: $$\varphi = \frac{1+\sqrt{5}}{2}$$

The number $\varphi \approx 1.618\ldots$ is called the golden ratio.

The golden ratio: a broader perspective

- Studied since antiquity
- First defined by Euclid (*Elements*, c. 300 BC)
- Associated with perceptions of beauty
- Applications in art and architecture

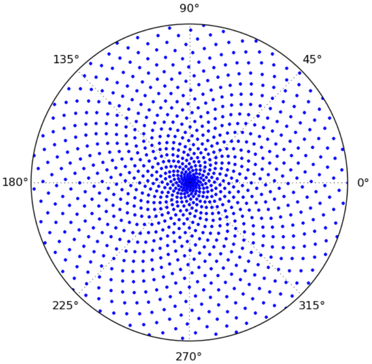


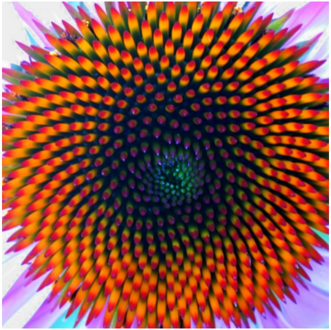


The golden ratio in plant growth

$$\alpha = 1/\varphi$$

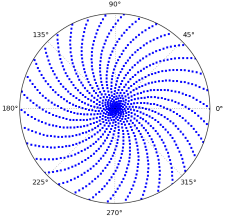
The golden ratio in plant growth



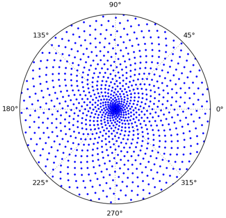


The golden ratio in plant growth

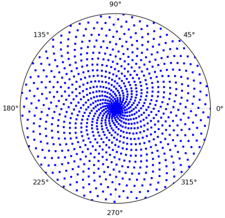
$\alpha = 222.4^\circ$



$\alpha = 1/\varphi \approx 222.5^\circ$



$\alpha = 222.6^\circ$



Nature seems to have found φ quite precisely!

Some properties of irrational numbers

Theorem

Every irrational number can be written as a continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

or, for short, $[a_0; a_1, a_2, \dots]$, where the a_i are positive integers.

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$$\begin{aligned}
\pi &= [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ \ldots] \\
e &= [2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ \ldots] \\
\sqrt{2} &= [1;\ 2,\ 2,\ 2,\ \ldots] \\
\varphi &= [1;\ 1,\ 1,\ 1,\ \ldots]
\end{aligned}$$

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The truncations

$$[a_0] = \frac{a_0}{1}, \quad [a_0; a_1] = \frac{a_1 a_0 + 1}{a_1},$$

$$[a_0; a_1, a_2] = \frac{a_2 (a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}, \quad \ldots$$

give a sequence of rational approximations called convergents.

Some properties of irrational numbers

Theorem

The convergent $[a_0; a_1, a_2, \ldots, a_k] \equiv m/n$ provides the best approximation among all rationals m'/n' with $n' \leq n$.

The truncations

$$[a_0] = \frac{a_0}{1}, \quad [a_0; a_1] = \frac{a_1 a_0 + 1}{a_1},$$

$$[a_0; a_1, a_2] = \frac{a_2\,(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}, \quad \ldots$$

give a sequence of rational approximations called convergents.

The most irrational number

A few convergents:

- π: $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}$
- e: $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}$
- $\sqrt{2}$: $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$

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$$22/7 \approx 3.14285714\ldots$$
$$333/106 \approx 3.14150943\ldots$$
$$355/113 \approx 3.14159292\ldots$$

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- e: $2, \quad 3, \quad \frac{8}{3}, \quad \frac{11}{4}, \quad \frac{19}{7}$
- $\sqrt{2}$: $1, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}$

$$22/7 \approx 3.14285714\ldots$$
$$333/106 \approx 3.14150943\ldots$$
$$355/113 \approx 3.14159292\ldots$$

What makes a number easy to approximate rationally?

The most irrational number

A few convergents:

- π: $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}$
- e: $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}$
- $\sqrt{2}$: $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

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Large denominators mean small numbers!

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The golden ratio has the slowest converging representation.

Back to the Fibonacci numbers

Theorem

The ratio of successive Fibonacci numbers $F_{n+1}/F_n \to \varphi$ as $n \to \infty$.

Back to the Fibonacci numbers

Theorem (in English)

The ratio of successive Fibonacci numbers $F_{n+1}/F_n \approx \varphi$, and the approximation gets better the bigger n is.

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Informally:

1. Exponential growth: $F_{n+1}/F_n \approx \theta$
2. Recurrence relation: $F_n = F_{n-1} + F_{n-2}$
3. Divide and rewrite: $\frac{F_n}{F_{n-1}} \frac{F_{n-1}}{F_{n-2}} = \frac{F_{n-1}}{F_{n-2}} + 1$
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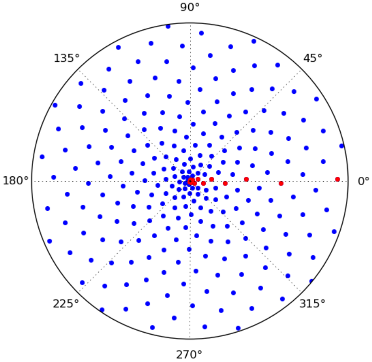
This is just the equation for the golden ratio, so $\theta \approx \varphi$.

Going around

$$\alpha = 1/\varphi \approx F_n/F_{n+1}$$



- F_n revolutions over F_{n+1} seeds
- No exact repeat since irrational
- Alternately overshoot and undershoot

Going around

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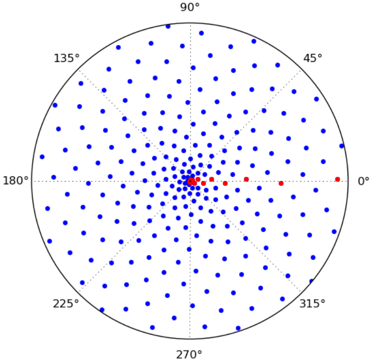
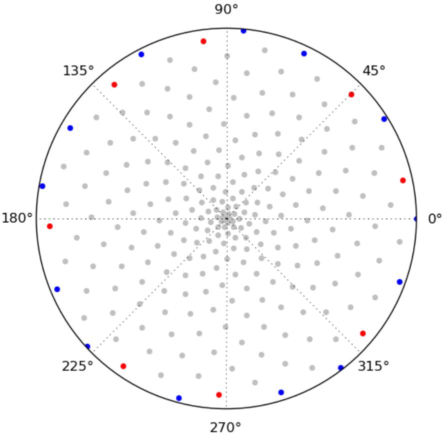


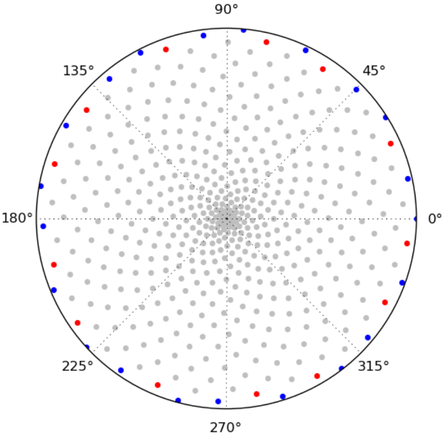
seeds	position
2	+0.236068
3	−0.145898
5	+0.090170
8	−0.055728
13	+0.034442
21	−0.021286
34	+0.013156
55	−0.008131
89	+0.005025
144	−0.003106

Origin of the spirals



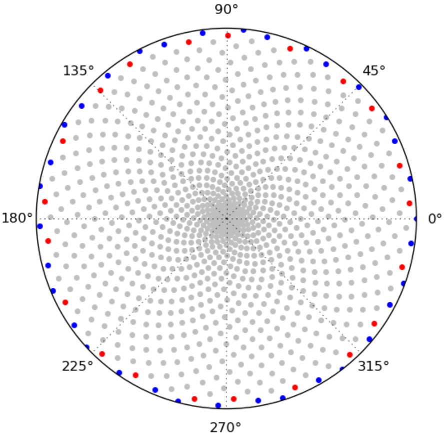
- Seed heads: 250
- CW spirals: 13
- CCW spirals: 13 + 8

Origin of the spirals



- Seed heads: 500
- CW spirals: 21 + 13
- CCW spirals: 21

Origin of the spirals



- Seed heads: 1000
- CW spirals: 34
- CCW spirals: 34 + 21

Origin of the spirals

Summary

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Final note

There is a very good reason why the Fibonacci numbers show up in at least one aspect of nature (plant growth)—and now you know what it is. (Spread the word!)

Questions?

MoMA (Sep 2008)